Analysis of the Cahn-Hilliard equation
with a chemical potential dependent mobility*

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Abstract

The aim of this paper is to study the well-posedness and the existence of global attractors for a family of Cahn-Hilliard equations with a mobility depending on the chemical potential. Such models arise from generalizations of the (classical) Cahn-Hilliard equation due to M. E. GURTIN.

1 Introduction

In this paper, we address the initial and boundary value problem for the following *generalized Cahn-Hilliard equation*:

$$\chi_t - \Delta\alpha\left(\delta\chi_t - \Delta\chi + \phi(\chi)\right) = 0 \qquad \text{in } \Omega \times (0,T), \tag{1.1}$$

where $\delta \geq 0$, $\Omega \subset \mathbb{R}^3$ is a bounded domain, $T > 0$ a finite time horizon, and $\alpha : \mathbb{R} \to \mathbb{R}$ a strictly increasing function.

The classical Cahn-Hilliard equation reads

$$\chi_t - \Delta w = 0, \quad w = -\Delta\chi + \phi(\chi) \qquad \text{in } \Omega \times (0,T),$$

where χ is the order parameter (corresponding to a density of atoms), w is the chemical potential (defined as a variational derivative of the free energy with respect to the order parameter), and ϕ is the derivative of a double-well potential. This equation plays an essential role in materials science and describes phase separation processes in binary alloys (see, e.g., [6, 7, 16]).

By considering a mechanical version of the second law of thermodynamics and introducing a new balance law for interactions at a microscopic level, M. E. GURTIN proposed in [10] the following equations:

$$\begin{cases} \chi_t - \text{div}(A(\chi, \nabla\chi, \chi_t, w)\nabla w) = 0, \\ w = \delta(\chi, \nabla\chi, \chi_t, w)\chi_t - \Delta\chi + \phi(\chi) \end{cases} \qquad \text{in } \Omega \times (0,T),$$

Taking δ constant and $A = a(w)I$, with $a : \mathbb{R} \to \mathbb{R}$ a positive function, we then obtain an equation of the form (1.1), in which α is some primitive of the function a.

In the viscous case $\delta > 0$, such equations have been studied in [18, 19]. Therein, results on the well-posedness and the existence of global attractors have been obtained.

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Our main aim in this paper is to treat the case $\delta = 0$. We also consider the viscous case $\delta > 0$ under different (and more general) assumptions on α and ϕ from those in [18, 19]. In particular, we prove the existence of solutions both in the non-viscous case $\delta = 0$ (cf. Theorem 1) and in the viscous case $\delta > 0$ (see Theorem 2). In the latter setting, under more restrictive assumptions on the nonlinearities α and ϕ, we also obtain (cf. Theorem 3.1) well-posedness and continuous dependence results for (the Cauchy problem for) (1.1). For $\delta > 0$ we are also able to study the asymptotic behavior of the system and establish the existence of the global attractor (see Theorem 3) in a quite general frame of assumptions on α and ϕ, which may allow for non-uniqueness of solutions. That is why, for this long-time analysis we rely on the notion of generalized semiflows proposed by J.M. BALL in [2], and on the extension given in [20]. Finally, relying on the short-trajectory approach developed in [13], we also conclude the existence of exponential attractors and, thus, of finite-dimensional global attractors. We recall that an exponential attractor is a compact and semi-invariant set which has finite fractal dimension and attracts the trajectories exponentially fast; note that the global attractor may attract the trajectories at a slow (polynomial) rate (see, e.g., [1, 9, 15]).

This paper is organized as follows. In Section 2, we define our notation and give some preliminary results. Then, in Section 3, we state our main results, whose proofs are carried out in the remaining sections. Finally, in Appendix, we introduce the approximation scheme for our problem and justify the a priori estimates (formally) developed throughout the paper.

2   Preliminaries

Notation and functional setup.   Throughout the paper, we consider a bounded domain $\Omega \subset \mathbb{R}^3$, with sufficiently smooth boundary $\partial\Omega$, and write $|\mathcal{O}|$ for the Lebesgue measure of any (measurable) subset $\mathcal{O} \subset \Omega$. Furthermore, given a Banach space B, we denote by $\|\cdot\|_B$ the norm in B and by ${}_{B'}\langle \cdot, \cdot \rangle_B$ the duality pairing between B' and B. We use the notation

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad Z := \left\{ v \in H^2(\Omega) \, : \, \partial_n v = 0 \right\},$$

and identify H with its dual space H', so that $Z \subset V \subset H \subset V' \subset Z'$, with dense and compact embeddings. We denote by $\mathcal{H}$, $\mathcal{V}$, $\mathcal{Z}$, $\mathcal{V}'$, and $\mathcal{Z}'$ the subspaces of the elements v of H, V, Z, V', and Z', respectively, with zero mean value $m(v) = \frac{1}{|\Omega|} {}_{Z'}\langle v, 1 \rangle_Z$. We consider the operator

$$A : V \to V', \qquad {}_{V'}\langle Au, v \rangle_V := \int_\Omega \nabla u \cdot \nabla v \quad \forall u, v \in V, \tag{2.1}$$

and note that $Au \in \mathcal{V}'$ for every $u \in V$. Indeed, the restriction of A to $\mathcal{V}$ is an isomorphism, so that we can introduce its inverse operator $\mathcal{N} : \mathcal{V}' \to \mathcal{V}$. We recall the relations

$$ {}_{V'}\langle Au, \mathcal{N}(v) \rangle_V = {}_{V'}\langle v, u \rangle_V \quad \forall u \in V, \ \forall v \in \mathcal{V}', \tag{2.2}$$

$$ {}_{V'}\langle u, \mathcal{N}(v) \rangle_V = \int_\Omega \nabla(\mathcal{N}(u)) \cdot \nabla(\mathcal{N}(v)) \, \mathrm{d}x = {}_{V'}\langle v, \mathcal{N}(u) \rangle_V \quad \forall u, v \in \mathcal{V}', \tag{2.3}$$

and that, on account of Poincaré's inequality for zero mean value functions, the following norms on V and V':

$$\|u\|_V^2 := {}_{V'}\langle Au, u \rangle_V + m(u)^2 \quad \forall u \in V,$$
$$\|v\|_{V'}^2 := {}_{V'}\langle v, \mathcal{N}(v - m(v)) \rangle_V + m(v)^2 \quad \forall v \in V',$$

are equivalent to the standard ones. It follows from the above formulae that

$$\|v\|_{V'}^2 = {}_{V'}\langle v, \mathcal{N}(v) \rangle_V = \|\mathcal{N}(v)\|_V^2 \quad \forall v \in \mathcal{V}'.$$

It is well known that the operator A (2.1) extends to an operator (which will be denoted by the same symbol) $A : H \to \mathcal{Z}'$. The inverse of the restriction of A to $\mathcal{H}$ is the extension of $\mathcal{N}$ to an operator $\mathcal{N} : \mathcal{Z}' \to \mathcal{H}$. By means of the latter, we define the space

$$\mathcal{W}^{-2,q}(\Omega) := \{ v \in \mathcal{Z}' \, : \, \mathcal{N}(v) \in L^q(\Omega) \} \qquad \text{for a given } q > 1,$$
$$\text{with the norm } \|v\|_{\mathcal{W}^{-2,q}(\Omega)} := \|\mathcal{N}(v)\|_{L^q(\Omega)}. \tag{2.4}$$

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The following result shows that, for $q \in (2,6)$ (which is the index range relevant to the analysis to be developed in what follows, cf. (3.7)), the space $\mathcal{W}^{-2,q}(\Omega)$ can be identified with the dual of the space

$$\mathcal{W}^{2,q'}(\Omega) = \left\{ z \in \mathcal{V} \,:\, Az \in L^{q'}(\Omega) \right\},$$

q' being the conjugate exponent of q. We endow the latter space with the norm $\|z\|_{\mathcal{W}^{2,q'}(\Omega)} := \|Az\|_{L^{q'}(\Omega)}$, which is equivalent to the standard $W^{2,q'}$-norm by the (generalized) Poincaré inequality.

Lemma 2.1. *For $q \in (2,6)$, the operator $\mathrm{J}: \mathcal{W}^{-2,q}(\Omega) \to (\mathcal{W}^{2,q'}(\Omega))'$ defined by*

$$_{(\mathcal{W}^{2,q'}(\Omega))'}\langle \mathrm{J}(v), z\rangle_{\mathcal{W}^{2,q'}(\Omega)} := {}_{V'}\langle Az, \mathcal{N}(v)\rangle_V \qquad \text{for all } z \in \mathcal{W}^{2,q'}(\Omega) \text{ and } v \in \mathcal{W}^{-2,q}(\Omega) \tag{2.5}$$

is an isomorphism.

Proof. We preliminarily note that, since $q \in (2,6)$, the conjugate exponent q' belongs to $(6/5, 2)$ and, consequently, one has the following embeddings:

$$\mathcal{H} \subset \mathcal{V}' \subset \mathcal{W}^{-2,q}(\Omega), \qquad L^{q'}(\Omega) \subset \mathcal{V}', \qquad \mathcal{Z} \subset \mathcal{W}^{2,q'}(\Omega). \tag{2.6}$$

Clearly, the operator J is well defined, linear, and continuous, since, for all $z \in \mathcal{W}^{2,q'}(\Omega)$ and $v \in \mathcal{W}^{-2,q}(\Omega)$,

$$\left| {}_{(\mathcal{W}^{2,q'}(\Omega))'}\langle \mathrm{J}(v), z\rangle_{\mathcal{W}^{2,q'}(\Omega)} \right| \leq \|Az\|_{L^{q'}(\Omega)} \|\mathcal{N}(v)\|_{L^q(\Omega)} \leq \|z\|_{\mathcal{W}^{2,q'}(\Omega)} \|v\|_{\mathcal{W}^{-2,q}(\Omega)}. \tag{2.7}$$

Furthermore, for every $v \in \mathcal{W}^{-2,q}(\Omega)$, one can choose $z_v = \mathcal{N}(|\mathcal{N}(v)|^{q-2}\mathcal{N}(v))$ (note that z_v is well defined and belongs to $\mathcal{V}$, since $|\mathcal{N}(v)|^{q-2}\mathcal{N}(v) \in L^{q'}(\Omega) \subset \mathcal{V}'$ by the second of (2.6)). Then,

$$_{(\mathcal{W}^{2,q'}(\Omega))'}\langle \mathrm{J}(v), z_v\rangle_{\mathcal{W}^{2,q'}(\Omega)} = \|\mathcal{N}(v)\|^q_{L^q(\Omega)},$$

$$\|Az_v\|_{L^{q'}(\Omega)} = \||\mathcal{N}(v)|^{q-2}\mathcal{N}(v)\|_{L^{q'}(\Omega)} = \|\mathcal{N}(v)\|^{q-1}_{L^q(\Omega)},$$

so that

$$\|\mathrm{J}(v)\|_{(\mathcal{W}^{2,q'}(\Omega))'} \geq \frac{\left| {}_{(\mathcal{W}^{2,q'}(\Omega))'}\langle \mathrm{J}(v), z_v\rangle_{\mathcal{W}^{2,q'}(\Omega)} \right|}{\|z_v\|_{\mathcal{W}^{2,q'}(\Omega)}} = \frac{\|\mathcal{N}(v)\|^q_{L^q(\Omega)}}{\|\mathcal{N}(v)\|^{q-1}_{L^q(\Omega)}} = \|\mathcal{N}(v)\|_{L^q(\Omega)} = \|v\|_{\mathcal{W}^{-2,q}(\Omega)}.$$

In view of (2.7), we conclude that J is an isometry. In particular, it is injective and the image $\mathrm{J}(\mathcal{W}^{-2,q}(\Omega))$ is closed in $(\mathcal{W}^{2,q'}(\Omega))'$. To conclude that J is surjective, we will prove that

$$\mathrm{J}(\mathcal{W}^{-2,q}(\Omega)) \text{ is dense in } (\mathcal{W}^{2,q'}(\Omega))'. \tag{2.8}$$

Indeed, let $\bar{z} \in \mathcal{W}^{2,q'}(\Omega)$ be such that

$$_{(\mathcal{W}^{2,q'}(\Omega))'}\langle \mathrm{J}(v), \bar{z}\rangle_{\mathcal{W}^{2,q'}(\Omega)} = 0 \text{ for all } v \in \mathcal{W}^{-2,q}(\Omega). \tag{2.9}$$

In particular, (2.9) holds for all $v \in \mathcal{H}$, so that, also in view of (2.2),

$$0 = {}_{V'}\langle A\bar{z}, \mathcal{N}(v)\rangle_V = {}_{V'}\langle v, \bar{z}\rangle_V = \int_\Omega \bar{z} v \quad \text{for all } v \in \mathcal{H}.$$

From the above relation, we easily conclude that $\bar{z} = 0$, whence (2.8). $\qquad\square$

A generalization of Poincaré's inequality. The following result will play an important role in the derivation of the *a priori estimates* of Section 4.1.

Lemma 2.2. *Let X and Y be Banach spaces, with X reflexive, and assume that*

$$X \Subset Y \text{ with compact embedding.} \tag{2.10}$$

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Consider

$$G : X \to Y \quad \text{a linear, weakly-weakly continuous functional,} \tag{2.11}$$

$$\Psi : X \to [0, +\infty) \quad \text{a 1-positively homogeneous,}$$
$$\text{sequentially weakly lower-semicontinuous functional.} \tag{2.12}$$

Assume that G and Ψ comply with the following compatibility condition*: for all $v \in X$,*

$$Gv = 0 \quad and \quad \Psi(v) = 0 \ \Rightarrow \ v = 0, \tag{2.13}$$

and that

$$\exists\, C \geq 1 : \quad \forall\, v \in X \quad \frac{1}{C} \left(\|v\|_Y + \|Gv\|_Y \right) \leq \|v\|_X \leq C \left(\|v\|_Y + \|Gv\|_Y \right). \tag{2.14}$$

Then,

$$\exists\, K > 0 : \quad \forall\, v \in X \quad \|v\|_X \leq K \left(\|Gv\|_Y + \Psi(v) \right). \tag{2.15}$$

Proof. Assume, by contradiction, that (2.15) does not hold: then, there exists a sequence $\{v_n\} \subset X$ such that, for every $n \in \mathbb{N}$,

$$\|v_n\|_X > n \left(\|Gv_n\|_Y + \Psi(v_n) \right). \tag{2.16}$$

In particular, this yields that $\|v_n\|_X \neq 0$ for all n. Letting $w_n := v_n / \|v_n\|_X$ and using the 1-homogeneity of Ψ, we deduce from (2.16) that

$$\|Gw_n\|_Y + \Psi(w_n) < \frac{1}{n} \quad \text{for every } n \in \mathbb{N},$$

giving

$$Gw_n \to 0 \text{ in } Y \quad \text{and} \quad \Psi(w_n) \to 0 \text{ as } n \to +\infty. \tag{2.17}$$

On the other hand, by the reflexivity of X, there exists a subsequence $\{w_{n_k}\}$ weakly converging in X to some $\bar{w}$. In view of (2.10)–(2.12), we find

$$w_{n_k} \to w \text{ in } Y, \qquad Gw_{n_k} \rightharpoonup Gw \text{ in } Y, \qquad \Psi(w) \leq \lim_{k \to +\infty} \Psi(w_{n_k}).$$

Hence, (2.17) yields that $Gw = 0$ and $\Psi(w) = 0$, so that, by (2.13), $w = 0$. Thus, by (2.14) and (2.17),

$$\lim_{k \to +\infty} \|w_{n_k}\|_X \leq C \lim_{k \to +\infty} \left(\|w_{n_k}\|_Y + \|Gw_{n_k}\|_Y \right) = 0,$$

in contrast with the fact that $\|w_n\|_X = 1$ for all $n \in \mathbb{N}$. $\qquad \square$

A compactness criterion. Let

$$\mathcal{O} \subset \mathbb{R}^d, \ d \geq 1, \text{ be an open set with } |\mathcal{O}| < +\infty,$$
$$B \text{ be a separable Banach space, and } 1 \leq p < +\infty. \tag{2.18}$$

We recall that a sequence $\{u_n\} \subset L^p(\mathcal{O}; B)$ is *p-uniformly integrable* (or simply *uniformly integrable* if $p = 1$) if

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 : \quad \forall\, J \subset \mathcal{O} \quad |J| < \delta \ \Rightarrow \ \sup_{n \in \mathbb{N}} \int_J \|u_n(y)\|_B^p \, \mathrm{d}y \leq \varepsilon. \tag{2.19}$$

We quote the following result (cf. [8, Thm. III.6]) which will be extensively used in what follows.

Theorem 2.1. *In the setting of (2.18), given a sequence $\{u_n\} \subset L^p(\mathcal{O}; B)$, assume that there exist a subsequence $\{u_{n_k}\}$ and a measurable function $u : \mathcal{O} \to B$ such that*

$$u_{n_k}(y) \to u(y) \quad in \ B \ for \ almost \ all \ y \in \mathcal{O}.$$

Then, $u_{n_k} \to u$ in $L^p(\mathcal{O}; B)$ if and only if it is p-uniformly integrable.

Finally, for the reader's convenience, here below we report the celebrated lower semicontinuity result due to A.D. IOFFE [12].

Theorem 2.2. *Let* $f : \mathcal{O} \times \mathbb{R}^n \times \mathbb{R}^m \to [0, +\infty]$, n, $m \geq 1$, *be a measurable non-negative function such that*

$$f(x, \cdot, \cdot) \quad \text{is lower semicontinuous on } \mathbb{R}^n \times \mathbb{R}^m \text{ for every } x \in \mathcal{O}, \tag{2.20}$$

$$f(x, u, \cdot) \quad \text{is convex on } \mathbb{R}^m \text{ for every } (x, u) \in \mathcal{O} \times \mathbb{R}^n. \tag{2.21}$$

Let (u_k, v_k), $(u, v) : \mathcal{O} \to \mathbb{R}^n \times \mathbb{R}^m$ *be measurable functions such that*

$$u_k(x) \to u(x) \quad \text{in measure in } \mathcal{O}, \qquad v_k \rightharpoonup v \quad \text{weakly in } L^1(\mathcal{O}; \mathbb{R}^m).$$

Then,

$$\liminf_{k \to +\infty} \int_{\mathcal{O}} f(x, u_k(x), v_k(x)) \, \mathrm{d}x \geq \int_{\mathcal{O}} f(x, u(x), v(x)) \, \mathrm{d}x. \tag{2.22}$$

2.1 Global attractors for generalized semiflows

As mentioned in the introduction, in order to study the long-time behavior of solutions to the generalized Cahn-Hilliard equation (1.1) *in the viscous case*, we rely on the theory of *generalized* semiflows introduced by J.M. BALL in [2]. In order to make this paper as self-contained as possible, in this section we recall the main definitions and results of this theory, closely following [2].

Notation 2.3. *The phase space is a (not necessarily complete) metric space* (X, d_X), *the distance* d_X *inducing the Hausdorff semidistance* e *of two non-empty subsets* A, $B \subset X$ *by the formula* $\mathrm{e}(A, B) := \sup_{a \in A} \inf_{b \in B} \mathrm{d}_X(a, b)$.

Definition 2.4 (Generalized semiflow). *A generalized semiflow* $\mathcal{S}$ *on* X *is a family of maps* $g : [0, +\infty) \to X$ *(referred to as "solutions") satisfying the following properties:*

(P1) (Existence) *for any* $g_0 \in X$, *there exists at least one* $g \in \mathcal{S}$ *such that* $g(0) = g_0$;

(P2) (Translates of solutions are solutions) *for any* $g \in \mathcal{S}$ *and* $\tau \geq 0$, *the map* $g^\tau(t) := g(t + \tau)$, $t \in [0, +\infty)$, *belongs to* $\mathcal{S}$;

(P3) (Concatenation) *for any* g, $h \in \mathcal{S}$ *and* $\tau \geq 0$ *with* $h(0) = g(\tau)$, *then* $z \in \mathcal{S}$, z *being the map defined by*

$$z(t) := \begin{cases} g(t) & \text{if } 0 \leq t \leq \tau, \\ h(t - \tau) & \text{if } t > \tau; \end{cases} \tag{2.23}$$

(P4) (Upper-semicontinuity w.r.t. the initial data) *if* $\{g_n\} \subset \mathcal{S}$ *and* $g_n(0) \to g_0$, *then there exist a subsequence* $\{g_{n_k}\}$ *of* $\{g_n\}$ *and* $g \in \mathcal{S}$ *such that* $g(0) = g_0$ *and* $g_{n_k}(t) \to g(t)$ *for all* $t \geq 0$.

Orbits, ω-limits and attractors. Given a solution $g \in \mathcal{S}$, we recall that the ω-limit $\omega(g)$ of g is defined by

$$\omega(g) := \{x \in X \; : \; \exists \{t_n\} \subset [0, +\infty), \; t_n \to +\infty, \text{ such that } g(t_n) \to x\}.$$

Similarly, the ω-limit of a set $E \subset X$ is given by

$$\omega(E) := \big\{ x \in X \; : \exists \{g_n\} \subset \mathcal{S} \text{ such that } \{g_n(0)\} \subset E, \; \{g_n(0)\} \text{ is bounded, and}$$

$$\exists \{t_n\} \subset [0, +\infty), \; t_n \to +\infty, \text{ such that } g_n(t_n) \to x \big\}.$$

Furthermore, we say that $w : \mathbb{R} \to X$ is a *complete orbit* if, for any $s \in \mathbb{R}$, the translate map w^s, restricted to the positive half-line $[0, +\infty)$, belongs to $\mathcal{S}$. For every $t \geq 0$, we can introduce the operator $T(t) : 2^X \to 2^X$ by setting

$$T(t)E := \{g(t) \; : \; g \in \mathcal{S} \text{ with } g(0) \in E\} \quad \text{for all } E \subset X, \tag{2.24}$$

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and define, for $\tau \geq 0$, the set
$$\gamma^\tau(E) := \cup_{t \geq \tau} T(t)E \,.$$

The family of operators $\{T(t)\}_{t \geq 0}$ defines a *semigroup* on the power set 2^X. Given subsets $U, E \subset X$, we say that U *attracts* E if $\mathrm{e}(T(t)E, U) \to 0$ as $t \to +\infty$. Furthermore, we say that U is *fully invariant* if $T(t)U = U$ for every $t \geq 0$. Finally, a set $\mathcal{A} \subset X$ is the *global attractor* for $\mathcal{S}$ iff it is compact, fully invariant under $\mathcal{S}$, and attracts all the bounded sets of X.

Compactness and dissipativity properties. Let $\mathcal{S}$ be a generalized semiflow. We say that $\mathcal{S}$ is

  eventually bounded iff, for every bounded set $B \subset X$, there exists $\tau \geq 0$ such that $\gamma^\tau(B)$ is bounded;

  point dissipative iff there exists a bounded set $B_0 \subset X$ such that, for any $g \in \mathcal{S}$, there exists $\tau \geq 0$ such that $g(t) \in B_0$ for all $t \geq \tau$. The set B_0 is then called a (pointwise) *absorbing* set;

  compact iff, for any sequence $\{g_n\} \subset \mathcal{S}$ with $\{g_n(0)\}$ bounded, there exists a subsequence $\{g_{n_k}\}$ such that $\{g_{n_k}(t)\}$ is convergent for any $t > 0$.

We note that the notions that we have just introduced are not independent one from another (cf. [2, Props. 3.1 and 3.2] for more details).

Lyapunov function. The notion of a *Lyapunov function* can be introduced starting from the following definitions: we say that a complete orbit $g \in \mathcal{S}$ is *stationary* if there exists $x \in X$ such that $g(t) = x$ for all $t \in \mathbb{R}$ - such an x is then called a *rest point*. Note that the set of rest points of $\mathcal{S}$, denoted by $Z(\mathcal{S})$, is closed in view of **(P4)**. A function $V : X \to \mathbb{R}$ is said to be a *Lyapunov function* for $\mathcal{S}$ if V is continuous, $V(g(t)) \leq V(g(s))$ for all $g \in \mathcal{S}$ and $0 \leq s \leq t$ (i.e., V decreases along all solutions), and, whenever the map $t \mapsto V(g(t))$ is constant for some complete orbit g, then g is a stationary orbit.

Existence of the global attractor. The following theorem subsumes the main results from [2] (cf. Thms. 3.3, 5.1, and 6.1 therein) and provides the basic criteria for the existence of the global attractor $\mathcal{A}$ for a generalized semiflow $\mathcal{S}$.

Theorem 2.3. *Let $\mathcal{S}$ be an eventually bounded and compact generalized semiflow. Assume that $\mathcal{S}$ also admits a Lyapunov function V and that*

$$\text{the set of its rest points } Z(\mathcal{S}) \text{ is bounded.} \tag{2.25}$$

Then, $\mathcal{S}$ is also point dissipative, and, consequently, it possesses a global attractor. Moreover, the attractor $\mathcal{A}$ is unique, it is the maximal compact fully invariant subset of X, and it can be characterized as

$$\mathcal{A} = \bigcup\{\omega(B) \ : \ B \subset X \text{ bounded}\} = \omega(X). \tag{2.26}$$

Finally, for every $g \in \mathcal{S}$,

$$\omega(g) \subset Z(\mathcal{S}). \tag{2.27}$$

Remark 2.5. Actually, it is immediate to check that, if $\mathcal{S}$ is compact, eventually bounded, and admits a Lyapunov function, then condition (2.25) can be replaced by

$$\exists \mathcal{D} \subset X, \quad \mathcal{D} \neq \emptyset, \quad \text{such that} \quad \begin{cases} T(t)\mathcal{D} \subset \mathcal{D} \quad \forall t \geq 0, \\ \text{the set } Z(\mathcal{S}) \cap \mathcal{D} \text{ is bounded in } X. \end{cases} \tag{2.28}$$

Then, under these hypotheses, $\mathcal{S}$ also possesses a (unique) global attractor $\mathcal{A} \subset \mathcal{D}$ and (2.27) holds.

3 Main results

3.1 A global existence result for the non-viscous problem

Assumptions on the nonlinearities. We assume that

$$\alpha : \mathbb{R} \to \mathbb{R} \qquad \text{is a strictly increasing, differentiable function such that}$$
$$\exists\, p \geq 0, \;\; \exists\, C_1, C_2 > 0 : \quad \forall\, r \in \mathbb{R} \qquad C_1 \left(|r|^{2p} + 1 \right) \leq \alpha'(r) \leq C_2 \left(|r|^{2p} + 1 \right) . \tag{H1}$$

Clearly, the latter growth condition entails that

$$\exists\, C_3, C_4, C_5 > 0 : \quad \forall\, r \in \mathbb{R} \qquad C_3 |r|^{2p+1} - C_4 \leq \alpha(r)\,\mathrm{sign}(r) \leq C_5 \left(|r|^{2p+1} + 1 \right) . \tag{3.1}$$

Concerning the nonlinearity ϕ, we require that

$$\mathrm{dom}(\phi) = I, \quad I \text{ being an open, possibly unbounded, interval } (a,b), \; -\infty \leq a < 0 < b \leq +\infty,$$
$$\phi \in \mathrm{C}^1(I),$$
$$\lim_{r \searrow a} \phi(r) = -\infty, \qquad \lim_{r \nearrow b} \phi(r) = +\infty, \tag{H2}$$
$$\lim_{r \searrow a} \phi'(r) = \lim_{r \nearrow b} \phi'(r) = +\infty .$$

We shall denote by $\widehat{\phi}$ (one of) the antiderivative(s) of ϕ. It follows from the above assumptions that $\widehat{\phi}$ is bounded from below. Hereafter, for the sake of simplicity, we assume that

$$\widehat{\phi}(r) \geq 0 \quad \text{for all } r \in I. \tag{3.2}$$

Furthermore, (H2) obviously yields that

$$\exists\, C_{\phi,1} > 0 : \quad \forall\, r \in I \qquad \phi'(r) \geq -C_{\phi,1} , \tag{3.3}$$

namely, ϕ is a Lipschitz perturbation of a non-decreasing function. In particular, we will use the fact that there exists a non-decreasing function $\beta : I \to \mathbb{R}$ such that

$$\phi(r) = \beta(r) - C_{\phi,1} r \qquad \forall\, r \in I . \tag{3.4}$$

Consequently, $\widehat{\phi}$ is a quadratic perturbation of a convex function. Arguing in the very same way as in [14] (where the case $I = (-1,1)$ was considered), it can be proved that, under these conditions, the following crucial estimate holds:

$$\forall\, m \in (a,b) \;\; \exists\, C_m, C_m' > 0 : \quad \forall\, r \in (a-m, b-m) \quad |\phi(r+m)| \leq C_m \phi(r+m) r + C_m' . \tag{3.5}$$

Finally, we also assume that

$$\exists\, \sigma \in (0,1), \;\; \exists\, C_6 > 0 : \quad \forall\, r \in (a,b) \quad |\phi(r)|^\sigma \leq C_6 \left(\widehat{\phi}(r) + 1 \right) , \tag{H3}$$

and that the following *compatibility condition* holds between σ and the growth index p of α in (H1):

$$\sigma > \max \left\{ \frac{6p-3}{6p+2}, 0 \right\} . \tag{H4}$$

Hence, if $p \leq 1/2$, then any $\sigma \in (0,1)$ is admissible, while if, for instance, $p = 1$, then the range of admissible σ's is $(3/8, 1)$, and it is $(9/14, 1)$ for $p = 2$.

Notation 3.1. Hereafter, we will use, for every $p \geq 0$, the short-hand notation

$$\rho_p := \frac{2p+2}{2p+1}, \qquad \kappa_p := \frac{6p+6}{2p+1}, \qquad \eta_{p\sigma} = \frac{6-\sigma}{(3-3\sigma)(2p+1)} . \tag{3.6}$$

For later convenience, we note that ρ_p and κ_p are decreasing functions of p and

$$1 < \rho_p < 2, \qquad 3 < \kappa_p < 6 \qquad \text{for every } p \geq 0. \tag{3.7}$$

Furthermore, it can be checked that

$$\eta_{p\sigma} > 1 \quad \text{for every } p \geq 0 \text{ and for all } \sigma > \max \left\{ \frac{6p-3}{6p+2}, 0 \right\} . \tag{3.8}$$

The existence result. We are now able to give the variational formulation of the boundary value problem associated with (1.1) in the non-viscous case.

Problem 1. Find a pair (χ, w) fulfilling

$$\chi_t + A(\alpha(w)) = 0 \qquad \text{in } \mathcal{W}^{-2,\kappa_p}(\Omega) \quad \text{a.e. in } (0,T), \tag{3.9}$$

$$A\chi + \phi(\chi) = w \qquad \text{a.e. in } \Omega \times (0,T). \tag{3.10}$$

Note that, owing to Lemma 2.1, (3.9) is equivalent to

$$\begin{gathered} \mathcal{W}^{-2,\kappa_p}(\Omega)\langle \chi_t, v\rangle_{\mathcal{W}^{2,\kappa_{p'}}(\Omega)} + \mathcal{W}^{-2,\kappa_p}(\Omega)\langle A(\alpha(w)), v\rangle_{\mathcal{W}^{2,\kappa_{p'}}(\Omega)} = 0 \\ \text{for all } v \in \mathcal{W}^{2,\kappa_{p'}}(\Omega) \quad \text{a.e. in } (0,T). \end{gathered} \tag{3.11}$$

Theorem 1. *Under assumptions* (H1)–(H4)*, for every initial datum χ_0 satisfying*

$$\chi_0 \in V, \qquad \widehat{\phi}(\chi_0) \in L^1(\Omega), \tag{3.12}$$

there exists at least a solution (χ, w) to Problem 1, with the regularity

$$\chi \in L^2(0,T; W^{2,6}(\Omega)) \cap L^\infty(0,T; V), \qquad \chi_t \in L^{\eta_{p\sigma}}(0,T; \mathcal{W}^{-2,\kappa_p}(\Omega)), \tag{3.13}$$

$$w \in L^2(0,T; V), \qquad \alpha(w) \in L^{\eta_{p\sigma}}(0,T; L^{\kappa_p}(\Omega)), \tag{3.14}$$

fulfilling the initial condition

$$\chi(0) = \chi_0 \quad \text{in } V. \tag{3.15}$$

A formal proof of this result will be developed in Section 4 and rigorously justified in Appendix.

3.2 A global existence result for the viscous problem

We replace our assumptions (H2)–(H4) on ϕ and its antiderivative $\widehat{\phi}$ by

$$\begin{gathered} \widehat{\phi} : \mathbb{R} \to \mathbb{R} \quad \text{belongs to } \mathrm{C}^2(\mathbb{R}) \text{ and satisfies} \\ \exists C_7 > 0 : \quad \forall r \in \mathbb{R} \quad |\phi(r)| \le C_7\left(\widehat{\phi}(r) + 1\right). \end{gathered} \tag{H5}$$

The latter assumption means that we consider potentials with at most an exponential growth at ∞, and it clearly yields that $\widehat{\phi}$ is bounded from below. Hence, as in (3.2), we again assume that $\widehat{\phi}$ takes non-negative values. Furthermore, as in the non-viscous case we require that

$$\exists C_{\phi,2} > 0 : \quad \forall r \in \mathbb{R} \qquad \phi'(r) \ge -C_{\phi,2}. \tag{H6}$$

This and (H5) imply that the map

$$r \in \mathbb{R} \mapsto \widehat{\phi}(r) + \frac{C_{\phi,2}}{2}r^2 \quad \text{is convex and bounded from below.} \tag{3.16}$$

Remark 3.2. Let us point out that (3.16) yields

$$|\widehat{\phi}(r)| \le |\widehat{\phi}(0)| + |\phi(r)||r| + \frac{C_{\phi,2}}{2}r^2 \quad \text{for all } r \in \mathbb{R}. \tag{3.17}$$

Indeed, it follows from (3.16) and an elementary convexity inequality that, for every $r \in \mathbb{R}$,

$$\widehat{\phi}(0) - \widehat{\phi}(r) - \frac{C_{\phi,2}}{2}r^2 \ge -r\left(\phi(r) + C_{\phi,2}r\right),$$

whence we deduce (3.17) with straightforward algebraic manipulations.

We will address the analysis of the Cahn-Hilliard equation (1.1) in the viscous case under the aforementioned assumptions. The related variational formulation reads

Problem 2. Given $\delta > 0$, find a pair (χ, w) fulfilling

$$\chi_t + A(\alpha(w)) = 0 \qquad \text{a.e. in } \Omega \times (0,T), \tag{3.18}$$

$$\delta\chi_t + A\chi + \phi(\chi) = w \qquad \text{a.e. in } \Omega \times (0,T). \tag{3.19}$$

The existence result.

Theorem 2. *Assume* (H1), (H5), *and* (H6). *Then, for every initial datum* χ_0 *complying with* (3.12), *there exists at least a solution* (χ, w) *to Problem 2, with the regularity*

$$\chi \in L^2(0,T;Z) \cap L^\infty(0,T;V) \cap H^1(0,T;H), \tag{3.20}$$

$$w \in L^2(0,T;V) \cap L^{2p+2}(0,T;L^\infty(\Omega)), \qquad \alpha(w) \in L^{\rho_p}(0,T;Z), \tag{3.21}$$

and such that χ *satisfies the initial condition* (3.15).

We refer to Section 4 for a formal proof of Theorem 2 and to Appendix for all rigorous calculations.

In addition, we also have the following regularity result, which plays a key role in Section 3.4.

Proposition 3.3. *Assume* (H1), (H5), *and* (H6). *Assume that, in addition,* ϕ *satisfies*

$$\widehat{\phi} \in \mathrm{C}^2(\mathbb{R}) \quad and \quad \exists\, C_{\phi,3} > 0 : \quad \forall\, r \in \mathbb{R} \quad |\phi'(r)| \le C_{\phi,3}(1 + |r|^4). \tag{3.22}$$

Then, for all $0 < \tau < T$, *the pair* (χ, w) *has the further regularity*

$$\chi \in L^\infty(\tau,T;Z) \cap H^1(\tau,T;V), \tag{3.23}$$

$$\alpha(w) \in L^{\rho_p}(\tau,T;H^3(\Omega)). \tag{3.24}$$

In particular, if $\chi_0 \in Z$, *then the above properties hold for any* $\tau \in [0,T)$.

Remark 3.4. From the proof of Proposition 3.3, it is not difficult to recover a uniform estimate of the following form:

$$\|\chi\|_{L^\infty(\tau,T;Z) \cap H^1(\tau,T;V)} + \|\alpha(w)\|_{L^{\rho_p}(\tau,T;H^3(\Omega))} \le Q(\tau^{-1}, \|\chi_0\|_V), \tag{3.25}$$

where Q is a suitable function which is nondecreasing with respect to both arguments.

3.3 Well-posedness for the viscous problem

Continuous dependence on the initial data and uniqueness. We will prove uniqueness (and continuous dependence) results for Problem 2 under more restrictive assumptions on α and on the growth of the function ϕ. In particular, we are going to consider two sets of assumptions.

First, we will suppose that ϕ behaves like a polynomial of degree at most 3. For the sake of simplicity and without loss of generality, we will carry out our analysis in the case when ϕ is the derivative of the double-well potential $\widehat{\phi}(r) = (r^2 - 1)^2/4$. Furthermore, we will replace (H1) by

$$\alpha : \mathbb{R} \to \mathbb{R} \quad \text{is a strictly increasing and differentiable function such that}$$
$$\exists\, C_9, C_{10} > 0 : \quad \forall\, r \in \mathbb{R} \qquad C_9 \le \alpha'(r) \le C_{10}, \tag{H7}$$

and (H5)–(H6) by

$$\phi(r) = r^3 - r \qquad \forall\, r \in \mathbb{R}. \tag{H8}$$

Theorem 3.1. *Assume* (H7) *and* (H8). *Let* χ_0^1 *and* χ_0^2 *be two initial data for Problem 2 fulfilling* (3.12) *and set* $M_* := \max_{i=1,2}\{\|\chi_0^i\|_V\}$; *let* χ_i, $i = 1, 2$, *be the corresponding solutions. Then, for every* $\delta > 0$, *there exists a positive constant* S_δ, *also depending on*

$$M_*, \ T, \ |\Omega|, \ C_9, \ and \ C_{10}, \tag{3.26}$$

such that

$$\|\chi_1(t) - \chi_2(t)\|_V + \|\chi_1 - \chi_2\|_{H^1(0,t;H) \cap L^2(0,t;Z)} \le S_\delta \|\chi_0^1 - \chi_0^2\|_V \quad \forall\, t \in [0,T]. \tag{3.27}$$

Our second continuous dependence results holds in the more general frame of assumptions of Proposition 3.3, but for more regular initial data. Indeed, we have

Theorem 3.2. *Assume that* (H1) *holds for some* $p \in [0,1]$, *and that* ϕ *complies with* (H5), (H6), *and* (3.22). *Let* χ_0^1 *and* χ_0^2 *be two initial data for Problem 2 such that* $\chi_0^i \in Z$ *and* $\widehat{\phi}(\chi_0^i) \in L^1(\Omega)$ *for* $i = 1, 2$, *and let* χ_i, $i = 1, 2$, *be the corresponding solutions. Then, for every* $\delta > 0$, *there exists a positive constant* S_δ, *also depending on* T, $|\Omega|$, C_1, C_2 *and* $M^* := \max_{i=1,2}\{\|\chi_0^i\|_Z\}$, *such that estimate* (3.27) *holds for all* $t \in [0,T]$.

3.4 Global attractor and exponential attractors for the viscous problem

The *energy functional* associated with Problem 2 reads

$$\mathcal{E} : X \to \mathbb{R}, \quad \mathcal{E}(v) := \frac{1}{2} \int_\Omega |\nabla v|^2 + \int_\Omega \widehat{\phi}(v) \quad \text{for all } v \in X. \tag{3.28}$$

Consequently, we introduce the phase space (X, d_X) of energy bounded solutions, defined by

$$X = \left\{ v \in V \,:\, \widehat{\phi}(v) \in L^1(\Omega) \right\},$$
$$\mathrm{d}_X(v_1, v_2) = \|v_1 - v_2\|_{H^1(\Omega)} + \left\| \widehat{\phi}(v_1) - \widehat{\phi}(v_2) \right\|_{L^1(\Omega)} \quad \text{for all } v_1,\, v_2 \in X. \tag{3.29}$$

The following definition details the properties of the solutions to Problem 2 to which our long-time analysis will apply.

Definition 3.5. *We say that a function* $\chi : [0, +\infty) \to X$ *is a* solution to Problem 2 on $(0, +\infty)$ *if, for all* $T > 0$, χ *enjoys regularity* (3.20) *on the interval* $(0, T)$ *and there exists a function* w, *with regularity* (3.21) *for all* $T > 0$, *such that equations* (3.18)–(3.19) *hold almost everywhere on* $\Omega \times (0, +\infty)$. *We set*

$$\mathcal{S} = \{\chi : [0, +\infty) \to X \,:\, \chi \text{ is a solution to Problem 2 on } (0, +\infty)\}. \tag{3.30}$$

We assume that, besides (H1), α complies with the following condition, slightly stronger than (H1):

$$\exists \,\mathsf{c}_\alpha > 0, \quad \exists \,\Psi : \mathbb{R} \to [0, +\infty) \text{ convex} : \quad \forall r \in \mathbb{R} \quad \alpha(r) r - \mathsf{c}_\alpha |r|^{2p+2} = \Psi(r). \tag{H9}$$

Hence, our first result asserts that the solution set $\mathcal{S}$ is a *generalized semiflow* in the sense of Definition 2.4.

Proposition 3.6. *Assume* (H1), (H5)–(H6). *Then,*

1. *every* $\chi \in \mathcal{S}$ *(cf.* (3.30)*) complies with the* energy identity

$$\delta \int_s^t \int_\Omega |\chi_t|^2 + \int_s^t \int_\Omega \alpha'(w)|\nabla w|^2 + \mathcal{E}(\chi(t)) = \mathcal{E}(\chi(s)) \quad \text{for all } 0 \le s \le t, \tag{3.31}$$

   the function $w : (0, +\infty) \to V$ *being defined by* (3.19) *on* $\Omega \times (0, +\infty)$.

2. *Assume that* α *in addition complies with* (H9). *Then, the set* $\mathcal{S}$ *is a generalized semiflow in the phase space* (3.29), *and its elements are continuous functions from* $[0, +\infty)$ *onto* X.

We prove our main result on the long-time behavior of the solutions to Problem 2 under a further condition on ϕ, which in particular implies (and thus replaces) (H6), namely

$$\lim_{r \to +\infty} \phi(r) = +\infty, \qquad \lim_{r \to -\infty} \phi(r) = -\infty,$$
$$\lim_{r \to +\infty} \phi'(r) = \lim_{r \to -\infty} \phi'(r) = +\infty. \tag{H10}$$

Theorem 3. *Assume* (H1), (H5), (H9), *and* (H10). *For a given* $\mathrm{m}_0 > 0$, *denote by* $\mathcal{D}_{\mathrm{m}_0}$ *the set*

$$\mathcal{D}_{\mathrm{m}_0} = \{\chi \in X \,:\, |m(\chi)| \le \mathrm{m}_0\}. \tag{3.32}$$

Then, the semiflow $\mathcal{S}$ *possesses a unique global attractor* $\mathcal{A}$ *in* $\mathcal{D}_{\mathrm{m}_0}$, *given by*

$$\mathcal{A} := \bigcup \{\omega(D) \,:\, D \subset \mathcal{D}_{\mathrm{m}_0} \text{ bounded}\}. \tag{3.33}$$

Finally, we have the following enhanced regularity for the elements of the ω-*limit of every trajectory:*

$$\forall \, p \in [1, +\infty) \ \exists \, C_p > 0 : \quad \forall \chi \in \mathcal{S}, \quad \forall \bar{\chi} \in \omega(\chi) \quad \|\bar{\chi}\|_{W^{2,p}(\Omega)} + \|\widehat{\phi}(\bar{\chi})\|_{L^p(\Omega)} \le C_p. \tag{3.34}$$

Remark 3.7. Notice that, in the case

$$\widehat{\phi} \text{ is a polynomial of even degree } \mathsf{m} \geq 4, \text{ with a positive leading coefficient,} \qquad (3.35)$$

then conditions (H5) and (H10) are satisfied.

Remark 3.8 (Enhanced regularity of the global attractor)**.** In addition to hypotheses (H1), (H5), (H9), and (H10) of Theorem 3, assume that ϕ complies with (3.22). Then, the enhanced regularity estimate (3.23) holds for $\mathcal{X}$.

This regularity is reflected in the further regularity

$$\mathcal{A} \subset Z, \qquad (3.36)$$

for the global attractor $\mathcal{A}$, which holds provided that one works with the (smaller) set of solutions to Problem 2 arising from the approximation procedure which will be detailed in Appendix. In fact, the estimates leading to (3.23) can be rigorously justified only for this approximate problem, as we will see in the proof of Proposition 3.3, cf. Section 4.3. Now, the aforementioned family of "approximable" solutions to Problem 2 (see, e.g., [1, 20, 21, 22] for analogous constructions) complies with the properties defining a generalized semiflow, except for the concatenation axiom. This has motivated the introduction in [20, 22] of the (slightly more general) notion of *weak* global attractor, tailored to the *weak* generalized semiflows without the concatenation property. Hence, relying on the abstract results of [20, 22] and arguing as in the proof of Theorem 3, it is straightforward to prove that the semiflow associated with the approximable solutions to Problem 2 admits a *weak* global attractor for which (3.36) holds. On the other hand, Theorem 4 below shows that, under the stronger assumptions of Theorem 3.2, the semiflow possesses the standard global attractor $\mathcal{A}$ satisfying (3.36), namely, $\mathcal{A}$ is a compact and invariant set which attracts (in the V-metric) all bounded sets of initial data as time goes to infinity.

We conclude this section by showing that it is also possible to construct an exponential attractor through the short-trajectories approach developed in [13]. Let us first set for a given $\tau > 0$

$$X_\tau = L^2(0, \tau; V), \quad Y_\tau = \left\{ u \in L^2(0, \tau; Z) \, : \, u_t \in L^2(0, \tau; H) \right\}$$

and observe that Y_τ is compactly embedded in X_τ.

Under assumptions (H1), (H5), (H6), and (3.22), we know that, for any $\mathcal{X}_0 \in V$ and any $T > 0$, there exists a pair $(\mathcal{X}, w)$ which solves Problem 2 with the regularity (3.20), (3.21), (3.23), (3.24) (cf. Theorem 2 and Proposition 3.3). In particular, $\mathcal{X} \in Y_T$. In addition, thanks to (3.31) and arguing in the same way as in the forthcoming Section 4.1, it is not difficult to show that $\|\mathcal{X}\|_{Y_T}$ can be estimated uniformly with respect to $\|\mathcal{X}_0\|_V$, . The energy identity also entails the existence of a bounded set $B^0 \subset V$ such that, if $(\mathcal{X}, w)$ is a solution to Problem 2 with the aforementioned properties, then there exists $t_0 > 0$, only depending on $\|\mathcal{X}_0\|_V$, such that $\mathcal{X}(t) \in B^0$ for all $t \geq t_0$ and $\mathcal{X}(t) \in B^0$ for all $t \geq 0$ whenever $\mathcal{X}_0 \in B^0$ (see the proof of the eventual boundedness of $\mathcal{S}$ in Section 5.2). Let us now consider the set $\mathcal{X}_\ell = \{\mathcal{X} : (0, \ell) \to V\}$ of all the ℓ-trajectories $\mathcal{X}$ such that $(\mathcal{X}, w)$ is a solution to Problem 2 satisfying (3.20), (3.21), (3.23), (3.24). Then, we endow this set with the X_ℓ-topology (note that it might be a non-complete metric space). Moreover, denoting by V_w the space V endowed with the weak topology, we have $\mathcal{X}_\ell \subset C^0([0, \ell]; V_w)$. Consequently, any ℓ-trajectory makes sense pointwise.

From now on, we assume that assumption (H1) holds for some $p \in [0, 1]$. Thanks to (3.23), for any ℓ-trajectory, there exists $\tau \in (0, \ell)$ such that $\mathcal{X}(\tau) \in Z$. This is sufficient to conclude that $\mathcal{X}$ is unique from τ on, as a consequence of Proposition 3.3 and Theorem 3.2. Therefore, if $\mathcal{X} \in \mathcal{X}_\ell$ and $T > \ell$, then there exists a unique $\tilde{\mathcal{X}} \in \mathcal{X}_T$ such that $\tilde{\mathcal{X}}|_{[0,\ell]} = \mathcal{X}$. Thus, we can define a semigroup L_t on $\mathcal{X}_\ell$ by setting

$$(L_t\mathcal{X})(\tau) := \tilde{\mathcal{X}}(t + \tau), \qquad \tau \in [0, \ell],$$

where $\tilde{\mathcal{X}}$ is the unique element of $\mathcal{X}_{\ell+\tau}$ such that $\tilde{\mathcal{X}}|_{[0,\ell]} = \mathcal{X}$.

Let us now set

$$B_\ell^0 := \left\{ \mathcal{X} \in \mathcal{X}_\ell \, : \, \mathcal{X}(0) \in B^0 \right\}.$$

Then, by Proposition 3.3, we can infer that the set $\left\{ \mathcal{X}|_{[\ell/2,\ell]} : \mathcal{X} \in B_\ell^0 \right\}$ is bounded in $L^\infty(\ell/2, \ell; Z)$. Hence, we can prove a continuous dependence estimate like (3.27), which allows us to apply [13, Lemma 2.1]

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and deduce that L_t is Lipschitz continuous on B_ℓ^0, uniformly with respect to $t \in [0, \tau]$ for any fixed $\tau > 0$. Observe that, arguing as in Section 4.3, we can prove that $B_\ell^1 = \overline{L_\tau(B_\ell^0)}^{X_\ell} \subseteq B_\ell^0$ for some $\tau > 0$. From this fact we deduce that the dynamical system (X_ℓ, L_t) has a global attractor $\mathcal{A}_\ell$ (see [13, Thm. 2.1]). In addition, $L_\tau : X_\ell \to Y_\ell$ is Lipschitz continuous for some $\tau > 0$. Indeed, recall that B_ℓ^1 is bounded in $L^\infty(0, \ell; Z) \cap H^1(0, \ell; V)$ and use (3.27). Thus, on account of [13, Thm. 2.2], we can infer that $\mathcal{A}_\ell$ has finite fractal dimension. In order to go back to the original geometric space V, we introduce the evaluation mapping $e : X_\ell \to V$, $e(\chi) := \chi(\ell)$. Then, we set $B^1 := e(B_\ell^1)$ and we note that, for any $\chi_0 \in B^1$, there is a unique solution to Problem 2, so that the solution operator S_t is well defined on B^1 and $S_t(B^1) \subseteq B^1$, for all $t \geq 0$. In addition, e is (Lipschitz) continuous on B_ℓ^1 (use (3.27) and [13, Lemma 2.1] once more). Therefore, we use [13, Thm. 2.4] to deduce that $\mathcal{A} := e(\mathcal{A}_\ell)$ is the finite-dimensional global attractor of the dynamical system (B^1, S_t).

It remains to prove the existence of an exponential attractor. We already know that L_t is Lipschitz continuous on B_ℓ^1, uniformly with respect to $t \in [0, \tau]$ for every fixed $\tau > 0$ (see above). Thus, we only need to show that $t \mapsto L_t \chi$ is Hölder continuous with values in V, uniformly with respect to $\chi \in B_\ell^1$. This follows from [13, Lemma 2.2], recalling that B_ℓ^1 is, in particular, bounded in $H^1(0, \ell; V)$. Hence, (X_ℓ, L_t) has an exponential attractor $\mathcal{E}_\ell$ and $\mathcal{E} := e(\mathcal{E}_\ell)$ is an exponential attractor for (B^1, S_t).

Summing up, we have proved the

Theorem 4. *Assume that* (H1) *holds for some* $p \in [0, 1]$. *Also, assume* (H5), (H6), *and* (3.22). *Then, there exists a bounded invariant set* $B^1 \subset V$ *such that Problem 2 generates a dynamical system* (B^1, S_t) *which possesses an exponential attractor* $\mathcal{E}$. *In addition, the system also has a global attractor* $\mathcal{A}$ *with finite fractal dimension.*

Note that, in the framework of Theorem 4, neither assumption (H9) nor (H10) are needed.

4 Proofs of Theorems 1 and 2

Scheme of the proofs of Theorems 1 and 2. We will prove Theorems 1 and 2 by taking the limit of a suitable approximation scheme for Problems 1 and 2. For the sake of readability, we postpone detailing such a scheme in Appendix.

In Section 4.1, we will instead perform all estimates leading to the aforementioned passage to the limit directly on systems (3.9)–(3.10) and (3.18)–(3.19). Note that, at this stage, some of the following calculations will only be formal, cf. Remark 4.2 below. Their rigorous justification will be given in Appendix, see Section A.1.

Next, in Section 4.2 (in Section 4.3, respectively), we will carry out a passage to the limit in some unspecified approximation scheme for Problem 1 (for Problem 2, respectively) and conclude the (formal) proof of Theorem 1 (of Theorem 2, respectively). In Section A.2, we will adapt the limiting arguments developed in Sections 4.2 and 4.3 to the approximation scheme for Problems 1 and 2 and carry out the rigorous proofs of the related existence theorems.

Notation 4.1. We will perform the a priori estimates on systems (3.9)–(3.10) and (3.18)–(3.19), distinguishing the ones which hold both in the viscous and the non-viscous cases from the ones which depend on the constant δ in (3.19) (which can be either strictly positive or equal to zero), and on our different assumptions on the nonlinearity ϕ in the viscous and non-viscous cases. Accordingly, we will use the generic notation C for most of the constants appearing in the forthcoming calculations and depending on the problem data, and C_δ (C_0, respectively) for those constants *substantially* depending on the problem data and on $\delta > 0$ (on $\delta = 0$, respectively). We will adopt the same convention for the constants S^i, S_δ^i, S_0^i, $i \geq 1$.

4.1 A priori estimates

First a priori estimate. We test (3.18) by w, (3.19) by χ_t, add the resulting relations, and integrate over some time interval $(0, t) \subset (0, T)$. Elementary calculations lead to

$$\int_0^t \int_\Omega \alpha'(w) |\nabla w|^2 + \delta \int_0^t \|\chi_t\|_H^2 + \frac{1}{2} \|\nabla \chi(t)\|_H^2 + \int_\Omega \widehat{\phi}(\chi(t)) = \frac{1}{2} \|\nabla \chi_0\|_H^2 + \int_\Omega \widehat{\phi}(\chi_0). \qquad (4.1)$$

Recalling (3.12), the second of (H1) (which, in particular, yields that α' is bounded from below on $\mathbb{R}$ by a positive constant) and the positivity of $\widehat{\phi}$ (cf. (3.2)), we conclude that, for some constant $S^1 > 0$,

$$\|\nabla w\|_{L^2(0,T;H)} + \|\nabla \chi\|_{L^\infty(0,T;H)} + \|\widehat{\phi}(\chi)\|_{L^\infty(0,T;L^1(\Omega))} \leq S^1. \tag{4.2}$$

First a priori estimate in the viscous case. In the case $\delta > 0$, from the previous a priori estimate we also have

$$\|\chi_t\|_{L^2(0,T;H)} + \|\chi\|_{L^\infty(0,T;V)} \leq S_\delta^1. \tag{4.3}$$

Second a priori estimate. We test (3.18) by 1 and find $m(\chi_t) = 0$ a.e. in $(0,T)$, so that, in particular,

$$m(\chi(t)) = m_0 := m(\chi_0) \qquad \forall\, t \in [0,T]. \tag{4.4}$$

Hence, testing (3.19) by 1, we obtain

$$m(\phi(\chi(t))) = m(w(t)) \qquad \text{for a.a. } t \in (0,T). \tag{4.5}$$

Second a priori estimate in the non-viscous case. It follows from (4.4) and the Poincaré inequality that

$$\|\chi\|_{L^\infty(0,T;V)} \leq S^2. \tag{4.6}$$

Third a priori estimate in the non-viscous case. We test (3.10) by $\chi - m(\chi)$: we have, for a.e. $t \in (0,T)$,

$$
\begin{aligned}
\|\nabla \chi(t)\|_H^2 + \int_\Omega \phi(\chi(t))\,(\chi(t) - m(\chi(t))) &= \int_\Omega w(t)\,(\chi(t) - m(\chi(t))) \\
&= \int_\Omega (w(t) - m(w(t)))\,(\chi(t) - m(\chi(t))) \\
&\leq C\|\nabla \chi(t)\|_H \|\nabla w(t)\|_H \leq C S^1 \|\nabla w(t)\|_H,
\end{aligned} \tag{4.7}
$$

the latter estimate ensuing from the Poincaré inequality for zero mean value functions and the previous (4.2). On the other hand, (3.5) and (4.4) yield that there exist constants $C_{m_0}, C'_{m_0} > 0$ such that, for a.e. $t \in [0,T]$,

$$\int_\Omega |\phi(\chi(t))| \leq C_{m_0} \int_\Omega \phi(\chi(t))\,(\chi(t) - m(\chi(t))) + C'_{m_0}. \tag{4.8}$$

Combining this with (4.7), we deduce that there exists $C > 0$, also depending on C_{m_0} and on C'_{m_0}, such that, for a.e. $t \in (0,T)$,

$$\int_\Omega |\phi(\chi(t))| \leq C\,(\|\nabla w(t)\|_H + 1). \tag{4.9}$$

Thus, in view of (4.2), we obtain an estimate for $\phi(\chi)$ in $L^2(0,T;L^1(\Omega))$. Finally, due to (4.5), we find

$$\|m(w)\|_{L^2(0,T)} \leq C_0.$$

Hence, by (4.2) and the Poincaré inequality, we conclude that

$$\|w\|_{L^2(0,T;V)} \leq S_0^1. \tag{4.10}$$

Third a priori estimate in the viscous case. Estimate (4.2) for $\widehat{\phi}(\chi)$ and (H5) yield that

$$\|\phi(\chi)\|_{L^\infty(0,T;L^1(\Omega))} \leq S_\delta^2. \tag{4.11}$$

Recalling (4.5), we immediately infer that

$$\|m(w)\|_{L^\infty(0,T)} \leq S_\delta^3, \tag{4.12}$$

whence, again,

$$\|w\|_{L^2(0,T;V)} \leq S_\delta^4. \tag{4.13}$$

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Fourth a priori estimate in the non-viscous case. We preliminarily observe that, thanks to (3.4), equation (3.10) can be rewritten as

$$A\chi + \beta(\chi) = w + C_{\phi,1}\chi \qquad \text{a.e. in } \Omega \times (0,T) \,. \tag{4.14}$$

Notice that, in view of (4.6) and (4.10), the right-hand side of (4.14) belongs to $L^2(0,T;L^6(\Omega))$. Hence, we can test (4.14) by $|\beta(\chi)|^4\beta(\chi)$ and easily conclude that

$$\|A\chi\|_{L^2(0,T;L^6(\Omega))} + \|\beta(\chi)\|_{L^2(0,T;L^6(\Omega))} \leq C_0 \,.$$

Then, also by standard elliptic regularity results, we find

$$\|\phi(\chi)\|_{L^2(0,T;L^6(\Omega))} + \|\chi\|_{L^2(0,T;W^{2,6}(\Omega))} \leq S_0^2 \,. \tag{4.15}$$

Fourth a priori estimate in the viscous case. We combine (4.3) and (4.13) and argue by comparison in (3.19). Relying on (H6) and on the related elliptic regularity estimate, we have

$$\|\phi(\chi)\|_{L^2(0,T;H)} \leq S_\delta^5 \,, \tag{4.16}$$

as well as an estimate for $A\chi$ in $L^2(0,T;H)$, so that

$$\|\chi\|_{L^2(0,T;Z)} \leq S_\delta^6 \,. \tag{4.17}$$

Fifth a priori estimate. It follows from (4.1) and (H1) that $\int_0^T \int_\Omega w^{2p}|\nabla w|^2 \leq C$, whence we conclude that

$$\|\nabla(|w|^p w)\|_{L^2(0,T;H)} \leq S^3 \,. \tag{4.18}$$

Sixth a priori estimate in the non-viscous case. From (4.2), (H3), and (4.15), we deduce that

$$\||\phi(\chi)|^\sigma\|_{L^{2/\sigma}(0,T;L^{6/\sigma}(\Omega)) \cap L^\infty(0,T;L^1(\Omega))} \leq C_0 \,. \tag{4.19}$$

Using the interpolation inequality

$$\forall\, v \in L^1(\Omega) \cap L^{6/\sigma}(\Omega) \quad \|v\|_{L^{1/\sigma}(\Omega)} \leq \|v\|_{L^1(\Omega)}^\theta \|v\|_{L^{6/\sigma}(\Omega)}^{1-\theta}, \quad \text{with } \theta = \frac{5\sigma}{6-\sigma},$$

we obtain the estimate

$$\||\phi(\chi)|^\sigma\|_{L^{q_\sigma}(0,T;L^{1/\sigma}(\Omega))} \leq C_0, \quad \text{with } q_\sigma = \frac{2}{\sigma}\frac{1}{1-\theta} = \frac{6-\sigma}{3\sigma - 3\sigma^2}, \tag{4.20}$$

whence a bound for $\phi(\chi)$ in $L^{\sigma q_\sigma}(0,T;L^1(\Omega))$. Taking into account (4.5), we conclude that

$$\|m(w)\|_{L^{\sigma q_\sigma}(0,T)} \leq C_0 \,, \quad \text{whence} \quad \||m(w)|^{p+1}\|_{L^{(\sigma q_\sigma)/(p+1)}(0,T)} \leq C_0 \,. \tag{4.21}$$

On the other hand, applying the *nonlinear* Poincaré inequality (2.15) with the choices $X = V$, $Y = H$, $Gv = \nabla v$, and $\Psi(v) = |\Omega|^{-p-1}|\int_\Omega |v|^{\frac{1}{p+1}}\text{sign}(v)|^{p+1}$, where $v = |w|^p w$, we find

$$\||w|^p w\|_V \leq K\left(\|\nabla(|w|^p w)\|_H + |m(w)|^{p+1}\right) \,. \tag{4.22}$$

Therefore, combining estimate (4.21) for $|m(w)|^{p+1}$ with (4.18), we finally obtain, owing to the Poincaré inequality (4.22),

$$\||w|^p w\|_{L^{(\sigma q_\sigma)/(p+1)}(0,T;V)} \leq C_0. \tag{4.23}$$

Using the embedding $V \subset L^6(\Omega)$ and the growth (3.1) for α, we infer

$$\|\alpha(w)\|_{L^{\eta_{p\sigma}}(0,T;L^{\kappa_p}(\Omega))} \leq S_0^3 \tag{4.24a}$$

(where we have used the fact that $(\sigma q_\sigma)/(2p+1)$ equals the index $\eta_{p\sigma}$ defined in (3.6)). Hence, by comparison in (3.9), we also conclude that

$$\|\chi_t\|_{L^{\eta_{p\sigma}}(0,T;\mathcal{W}^{-2,\kappa_p}(\Omega))} \leq S_0^4 \tag{4.24b}$$

(see again (3.6) for the definition of κ_p).

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Sixth a priori estimate in the viscous case. Combining (4.12), (4.18) and the Poincaré-type inequality (4.22), we deduce an estimate for $|w|^p w$ in $L^2(0, T; V)$. Then, arguing in the same way as for (4.24a), we have

$$\|\alpha(w)\|_{L^{\rho_p}(0,T;L^{\kappa_p}(\Omega))} \leq C_\delta, \tag{4.25}$$

the index ρ_p being defined in (3.6). Now, in view of estimate (4.3) for χ_t in $L^2(0, T; H)$, a comparison in (3.18) yields an estimate for $A(\alpha(w))$ in $L^2(0, T; H)$. By elliptic regularity results, we finally conclude that

$$\|\alpha(w)\|_{L^{\rho_p}(0,T;Z)} \leq S_\delta^7. \tag{4.26}$$

Seventh a priori estimate in the non-viscous case. Our aim is now to show that

$$\|\phi(\chi)\|_{L^{\sigma q_\sigma}(0,T;L^6(\Omega))} \leq S_0^5, \qquad \text{with } \sigma q_\sigma = \frac{6-\sigma}{3-3\sigma} > 2. \tag{4.27}$$

Indeed, again recalling the embedding $V \subset L^6(\Omega)$, we observe that (4.23) yields an estimate for w in $L^{\sigma q_\sigma}(0, T; L^{6p+6}(\Omega))$. Then, taking into account estimate (4.6) for χ in $L^\infty(0, T; L^6(\Omega))$, together with the aforementioned elliptic regularity argument, we find estimate (4.27) by a comparison in (4.14).

Seventh a priori estimate in the viscous case. We combine estimate (4.26), the continuous embedding $Z \subset L^\infty(\Omega)$, and the growth condition (3.1) to deduce an estimate for w in $L^{\rho_p(2p+1)}(0, T; L^\infty(\Omega))$, whence

$$\|w\|_{L^{2p+2}(0,T;L^\infty(\Omega))} \leq S_\delta^8. \tag{4.28}$$

Remark 4.2. Notice that all the a priori estimates for the viscous Problem 2 are in fact rigorously justified on system (3.18)–(3.19). This has significant repercussions on the long-time analysis of Problem 2. Indeed, this allows us to work with the semiflow associated with the solutions to Problem 2 (cf. (3.30)) and prove the existence of a global attractor in the sense of [2]. However, as pointed out in Remark 3.8, if we address further regularity properties of the attractor (e.g., (3.36)), then we need additional estimates which cannot be performed directly on system (3.18)–(3.19), due to insufficient regularity of the solutions. Thus, we have to rely on some approximation. On the one hand, this leads to a smoother attractor $\mathcal{A}$, but, on the other hand, we lose the concatenation property of the trajectories (cf. [20, 22]); moreover, only trajectories which are limits of the approximation scheme will be attracted by the smoother attractor $\mathcal{A}$.

We also point out that the viscous system (3.18)–(3.19) cannot be used as an approximation for the non-viscous problem. Indeed, it is not difficult to realize that the fourth a priori estimates (4.14)–(4.15) (yielding a bound for $\phi(\chi)$ which plays a crucial role in the ensuing calculations) are not compatible with the term $\delta\chi_t$ in (3.19).

This fact seems to suggest the use of two different approximation schemes for Problem 1 and Problem 2, which would lead to cumbersome and repetitious calculations. In order to circumvent this problem, we will construct in Appendix an approximation scheme depending on two distinct parameters and prove the existence of solutions to Problem 1 by passing to the limit in three steps. Since the (rigorous) proof of existence for Problem 2 can be performed along the very same lines, we have chosen not to detail it in Appendix.

4.2 Proof of Theorem 1

Let $\{(\chi_n, w_n)\}$ be some sequence of approximate solutions to Problem 1. Due to estimates (4.2), (4.6), (4.10), (4.15), and (4.24), applying standard compactness and weak compactness results (see [23]), we find that there exists a pair (χ, w) with the regularities specified by (3.13)–(3.14) such that, along a (not relabeled) subsequence, the following strong, weak, and weak* convergences hold as $n \to +\infty$:

$$\chi_n \to \chi \quad \text{in } L^2(0, T; W^{2-\varepsilon,6}(\Omega)) \cap L^q(0, T; V) \cap C^0([0, T]; H^{1-\varepsilon}(\Omega))$$
$$\text{for every } \varepsilon > 0 \text{ and } 1 \leq q < +\infty, \tag{4.29}$$

$$\chi_n \overset{*}{\rightharpoonup} \chi \quad \text{in } L^2(0, T; W^{2,6}(\Omega)) \cap L^\infty(0, T; V), \tag{4.30}$$

$$\chi_{n,t} \rightharpoonup \chi_t \quad \text{in } L^{\eta_{p\sigma}}(0, T; \mathcal{W}^{-2,\kappa_p}(\Omega)), \tag{4.31}$$

$$w_n \rightharpoonup w \quad \text{in } L^2(0, T; V). \tag{4.32}$$

Furthermore, there exists $\bar{\alpha} \in L^{\eta_{p\sigma}}(0,T; L^{\kappa_p}(\Omega))$ such that

$$\alpha(w_n) \rightharpoonup \bar{\alpha} \quad \text{in} \quad L^{\eta_{p\sigma}}(0,T; L^{\kappa_p}(\Omega)). \tag{4.33}$$

Now, estimate (4.27) in particular yields (recall that $\sigma q_\sigma > 2$) that

$$\text{the sequence } \{\phi(\chi_n)\} \text{ is uniformly integrable in } L^2(0,T;H). \tag{4.34}$$

Furthermore, we have, up to a further subsequence,

$$\phi(\chi_n(x,t)) \to \phi(\chi(x,t)) \qquad \text{for a.a.} (x,t) \in \Omega \times (0,T), \tag{4.35}$$

which is a consequence of the continuity of ϕ and of the pointwise convergence (up to a further subsequence) $\chi_n(x,t) \to \chi(x,t)$ a.e. in $\Omega \times (0,T)$ (cf. (4.29)). Combining (4.34) and (4.35) and recalling the compactness criterion Theorem 2.1, we conclude that

$$\phi(\chi_n) \to \phi(\chi) \qquad \text{in} \quad L^2(0,T;H). \tag{4.36}$$

Exploiting (4.29)–(4.36), one easily concludes that the triplet $(\chi, w, \bar{\alpha})$ satisfies

$$\chi_t + A\bar{\alpha} = 0 \qquad \text{in } \mathcal{W}^{-2,\kappa_p}(\Omega) \quad \text{a.e. in } (0,T), \tag{4.37}$$
$$A\chi + \phi(\chi) = w \qquad \text{a.e. in } \Omega \times (0,T). \tag{4.38}$$

Finally, in order to prove that

$$\bar{\alpha}(x,t) = \alpha(w(x,t)) \qquad \text{for a.a. } (x,t) \in \Omega \times (0,T), \tag{4.39}$$

we test the equation approximating (3.10) by w_n and integrate in time. We thus have

$$\lim_{n \to +\infty} \int_0^T \int_\Omega |w_n|^2 = \lim_{n \to +\infty} \int_0^T \int_\Omega \phi(\chi_n) w_n + \lim_{n \to +\infty} \int_0^T \int_\Omega \nabla \chi_n \cdot \nabla w_n$$
$$= \int_0^T \int_\Omega \phi(\chi) w + \int_0^T \int_\Omega \nabla \chi \cdot \nabla w$$
$$= \int_0^T \int_\Omega |w|^2,$$

where the second equality follows from convergences (4.29), (4.32), and (4.36), and the last one from (4.38). Hence, we conclude that

$$w_n \to w \quad \text{in } L^2(0,T;H), \quad \text{whence} \quad w_n \to w \text{ a.e. in } \Omega \times (0,T)$$

(the latter convergence holding up to a subsequence). By continuity of α, we also have $\alpha(w_n) \to \alpha(w)$ a.e. in $\Omega \times (0,T)$. Estimate (4.24a) (recall (3.8)) and again Theorem 2.1 yield, for instance, that

$$\alpha(w_n) \to \alpha(w) \quad \text{in } L^1(0,T; L^1(\Omega)),$$

whence the desired equality (4.39). $\qquad\square$

4.3 Proofs of Theorem 2 and Proposition 3.3

Proof of Theorem 2. Let $\{(\chi_n, w_n)\}$ be some sequence of approximate solutions to Problem 2. Thanks to estimates (4.2), (4.3), (4.13), (4.17), (4.26), and (4.28), applying standard compactness and weak compactness results (see [23]), we find a triplet $(\chi, w, \bar{\alpha})$ such that, along a (not relabeled) subsequence, the following strong, weak, and weak* convergences hold as $n \to +\infty$:

$$\chi_n \to \chi \quad \text{in} \quad L^2(0,T; H^{2-\varepsilon}(\Omega)) \cap L^q(0,T;V) \cap C^0([0,T]; H^{1-\varepsilon}(\Omega))$$
$$\text{for every } \varepsilon > 0 \text{ and } 1 \le q < +\infty, \tag{4.40}$$

$$\chi_n \xrightarrow{*} \chi \quad \text{in} \quad L^2(0,T;Z) \cap L^\infty(0,T;V) \cap H^1(0,T;H), \tag{4.41}$$

$$w_n \xrightarrow{*} w \quad \text{in} \quad L^2(0,T;V) \cap L^{2p+2}(0,T; L^\infty(\Omega)). \tag{4.42}$$

In particular, from (4.41), we deduce that

$$\mathcal{N}(\chi_{n,t}) \rightharpoonup \mathcal{N}(\chi_t) \qquad \text{in } L^2(0,T;Z). \tag{4.43}$$

Furthermore, by (4.25), there exists $\bar{\alpha} \in L^{\rho_p}(0,T;L^{\kappa_p}(\Omega))$ such that

$$\alpha(w_n) \rightharpoonup \bar{\alpha} \quad \text{in } L^{\rho_p}(0,T;L^{\kappa_p}(\Omega)). \tag{4.44}$$

Now, up to a subsequence, by the last of (4.40) and by continuity of ϕ, we have, for all $t \in [0,T]$,

$$\phi(\chi_n(\cdot,t)) \to \phi(\chi(\cdot,t)) \qquad \text{a.e. in } \Omega. \tag{4.45}$$

On the other hand, it follows from estimate (4.16) that

$$\text{the sequence } \{\phi(\chi_n)\} \text{ is uniformly integrable in } L^1(0,T;L^1(\Omega)). \tag{4.46}$$

Then, by (4.45)–(4.46) and Theorem 2.1, we conclude that, along the same subsequence as in (4.45), $\phi(\chi_n) \to \phi(\chi)$ in $L^1(0,T;L^1(\Omega))$. We then have, up to a subsequence,

$$\phi(\chi_n(t)) \to \phi(\chi(t)) \qquad \text{in } L^1(\Omega) \text{ for a.a. } t \in (0,T). \tag{4.47}$$

Next, using (4.11), we see that $\phi(\chi_n)$ is uniformly integrable in $L^\nu(0,T;L^1(\Omega))$ for all $\nu \in [1,+\infty)$. Applying Theorem 2.1, from (4.47), we deduce that

$$\phi(\chi_n) \to \phi(\chi) \qquad \text{in } L^\nu(0,T;L^1(\Omega)) \text{ for every } \nu \in [1,+\infty). \tag{4.48}$$

Collecting (4.40)–(4.44) and (4.48), we conclude that the triplet $(\chi, w, \bar{\alpha})$ satisfies

$$\chi_t + A\bar{\alpha} = 0 \qquad \text{a.e. in } \Omega \times (0,T), \tag{4.49}$$

$$\delta\chi_t + A\chi + \phi(\chi) = w \qquad \text{a.e. in } \Omega \times (0,T). \tag{4.50}$$

It remains to show that $\bar{\alpha} \equiv \alpha(w)$. To this aim, we note that α defines a maximal monotone graph in the duality $(L^{2p+2}(\Omega \times (0,T)), L^{\rho_p}(\Omega \times (0,T)))$ (note that ρ_p and $2p+2$ are conjugate exponents). Taking into account relations (4.42) and (4.44), and applying a well-known result from the theory of maximal monotone operators in Banach spaces (see [3, Lemma 1.3, p. 42]), it is then sufficient to prove that

$$\limsup_{n\to+\infty} \int_0^T \int_\Omega \alpha(w_n)w_n \le \int_0^T \int_\Omega \bar{\alpha}w. \tag{4.51}$$

Now,

$$\int_0^T \int_\Omega \alpha(w_n)w_n = \int_0^T \int_\Omega \big(\alpha(w_n) - m(\alpha(w_n))\big) w_n + |\Omega| \int_0^T m(\alpha(w_n)) m(w_n)$$
$$= -\int_0^T \int_\Omega w_n \mathcal{N}(\chi_{n,t}) + |\Omega| \int_0^T m(\alpha(w_n)) m(w_n), \tag{4.52}$$

where the second equality follows from (3.18). Then, using (2.2) and (3.19), we find the chain of inequalities

$$\liminf_{n\to+\infty} \Big(\int_0^T \int_\Omega w_n \mathcal{N}(\chi_{n,t}) \Big)$$
$$\ge \liminf_{n\to+\infty} \delta \int_0^T \|\chi_{n,t}\|_{V'}^2 + \lim_{n\to+\infty} \int_0^T \int_\Omega \chi_{n,t}\chi_n + \lim_{n\to+\infty} \int_0^T \int_\Omega \phi(\chi_n)\mathcal{N}(\chi_{n,t})$$
$$\ge \delta \int_0^T \|\chi_t\|_{V'}^2 + \int_0^T \int_\Omega \chi_t\chi + \int_0^T \int_\Omega \phi(\chi)\mathcal{N}(\chi_t)$$
$$= \int_0^T \int_\Omega w \mathcal{N}(\chi_t) = -\int_0^T \int_\Omega \big(\bar{\alpha} - m(\bar{\alpha})\big) w, \tag{4.53}$$

where the second inequality follows from convergences (4.40) and (4.41) for χ_n and from combining (4.43) with (4.48), while the subsequent identities are due to (4.49)–(4.50). On the other hand, it follows from (4.44) that

$$m(\alpha(w_n)) \rightharpoonup m(\bar\alpha) \qquad \text{in } L^{\rho_p}(0,T), \tag{4.54}$$

whereas, from (4.48), we gather that

$$m(w_n) = m(\phi(\chi_n)) \to m(\phi(\chi)) = m(w) \qquad \text{in } L^{2p+2}(0,T). \tag{4.55}$$

Combining (4.54)–(4.55), we conclude that

$$\lim_{n \to +\infty} |\Omega| \int_0^T m(\alpha(w_n))\, m(w_n) = |\Omega| \int_0^T m(\bar\alpha)\, m(w)\,. \tag{4.56}$$

Collecting (4.52), (4.53), and (4.56), we infer the desired (4.51). Ultimately, we have proved that

$$\alpha(w_n) \rightharpoonup \alpha(w) \ \text{ in } \ L^{\rho_p}(0,T;L^{\kappa_p}(\Omega)) \quad \text{and} \quad \lim_{n \to +\infty} \int_0^T \int_\Omega \alpha(w_n)w_n = \int_0^T \int_\Omega \alpha(w)w\,. \tag{4.57}$$

$\square$

Proof of Proposition 3.3. In order to prove that system (3.18)–(3.19) enjoys the regularization in time (3.23)–(3.24), using the Gagliardo-Nirenberg interpolation inequality we note that $L^2(0,T;Z) \cap L^\infty(0,T;V) \subset L^8(0,T;W^{1,12/5}(\Omega))$ with continuous embedding. Therefore, regularity (3.20) for χ and standard Sobolev embeddings yield

$$\|\chi\|_{L^8(0,T;L^{12}(\Omega))} \leq C\,. \tag{4.58}$$

Now, we test (3.19) by $A\chi_t$. Note that all the forthcoming computations are rigorous on the approximation scheme for Problem 2 which we will detail in Appendix. Elementary calculations yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_\Omega |A\chi|^2\right) + \delta \int_\Omega |\nabla\chi_t|^2 = I_1 + I_2\,, \tag{4.59}$$

with

$$I_1 := \int_\Omega \nabla w \cdot \nabla\chi_t \leq \frac{\delta}{2}\int_\Omega |\nabla\chi_t|^2 + \frac{1}{2\delta}\int_\Omega |\nabla w|^2\,, \tag{4.60}$$

$$\begin{aligned} I_2 &:= -\int_\Omega \phi'(\chi)\,(\nabla\chi \cdot \nabla\chi_t) \\ &\leq C_{\phi,3}\int_\Omega |\nabla\chi||\nabla\chi_t|\left(1 + |\chi|^4\right) \\ &\leq C\|\nabla\chi_t\|_H \|\nabla\chi\|_{L^6(\Omega)}\left(\|\chi\|^4_{L^{12}(\Omega)} + 1\right) \\ &\leq \frac{\delta}{4}\|\nabla\chi_t\|^2_H + C\left(\|\chi\|^8_{L^{12}(\Omega)} + 1\right)\|\chi\|^2_Z\,, \end{aligned} \tag{4.61}$$

where the second inequality follows from (3.22), the third one from the Hölder inequality, and the last one by taking into account the continuous embedding $Z \subset W^{1,6}(\Omega)$.

Collecting (4.59)–(4.60), taking into account (4.58), and applying the uniform Gronwall Lemma (see [24, Lemma III.1.1]), we find for every $\tau > 0$ an estimate of the form (3.25) for $\nabla\chi_t$ in $L^2(\tau,T;H)$ and for $A\chi$ in $L^\infty(\tau,T;H)$, whence

$$\chi \in L^\infty(\tau,T;Z) \cap H^1(\tau,T;V) \quad \text{for all } 0 < \tau < T\,.$$

Then, a comparison in (3.18) also yields a bound for $A(\alpha(w))$ in $L^{\rho_p}(\tau,T;V)$, whence an estimate for $\alpha(w)$ in $L^{\rho_p}(\tau,T;H^3(\Omega))$, in view of (3.21). Thus, we conclude (3.23)–(3.24), as well as estimate (3.25).

$\square$

5 Global attractor for Problem 2

5.1 Proof of Proposition 3.6

We need two preliminary lemmas. The first one clarifies some properties of the energy functional $\mathcal{E}$ (3.28).

Lemma 5.1. *Assume* (H5)–(H6). *Then, the functional* $\mathcal{E} : X \to \mathbb{R}$ *defined by* (3.28) *is bounded from below, lower-semicontinuous w.r.t. the H-topology, and satisfies the chain rule*

$$
\begin{aligned}
&\text{for all } v \in H^1(0,T;H) \text{ with } Av + \phi(v) \in L^2(0,T;H), \\
&\text{the map } t \in [0,T] \mapsto \mathcal{E}(v(t)) \text{ is absolutely continuous, and} \\
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(v(t)) = \int_\Omega v_t(t)\,(Av(t) + \phi(v(t))) \qquad \text{for a.a. } t \in (0,T)\,.
\end{aligned}
\tag{5.1}
$$

Proof. In order to prove the lower-semicontinuity property, we fix a sequence $\{v_n\}$ converging to some v in H and assume, without loss of generality, that $\sup_n \mathcal{E}(v_n) < +\infty$. Since $\widehat{\phi}$ is bounded from below, we conclude that $\{v_n\}$ is actually bounded in V, and thus $v_n \rightharpoonup v$ in V, yielding $\int_\Omega |\nabla v|^2 \le \liminf_n \int_\Omega |\nabla v_n|^2$. On the other hand,

$$
\begin{aligned}
\liminf_{n \to +\infty} \int_\Omega \widehat{\phi}(v_n) &= \liminf_{n \to +\infty} \int_\Omega \left(\widehat{\phi}(v_n) + \frac{C_{\phi,2}}{2}|v_n|^2 \right) - \frac{C_{\phi,2}}{2} \lim_{n \to +\infty} \int_\Omega |v_n|^2 \\
&\ge \int_\Omega \left(\widehat{\phi}(v) + \frac{C_{\phi,2}}{2}|v|^2 \right) - \frac{C_{\phi,2}}{2} \int_\Omega |v|^2\,,
\end{aligned}
$$

the latter inequality following from (3.16) and, for instance, from Ioffe's Theorem 2.2. Finally, to check the chain rule (5.1), we observe that the functional

$$
\mathcal{E}_{\mathrm{cv}}(v) := \mathcal{E}(v) + \frac{C_{\phi,2}}{2} \int_\Omega |v|^2 \qquad \text{for all } v \in X
\tag{5.2}
$$

is convex, thanks to (3.16). Then, (5.1) follows from the chain rule for $\mathcal{E}_{\mathrm{cv}}$, see [5, Lemma III.3.3]. $\qquad\square$

Lemma 5.2. *Assume* (H1). *Then,*

$$
\text{for all } w \in V \cap L^\infty(\Omega), \text{ there holds } \nabla\alpha(w(x)) = \alpha'(w(x))\nabla w(x) \text{ for a.a. } x \in \Omega\,.
\tag{5.3}
$$

Proof. Since Ω is smooth, we can take a sequence $\{w_k\} \subset \mathrm{C}^1(\overline{\Omega})$ such that $w_k \to w$ in $V \cap L^q(\Omega)$ for all $1 \le q < +\infty$. Clearly, for all $k \in \mathbb{N}$, there holds

$$
\nabla\alpha(w_k(x)) = \alpha'(w_k(x))\nabla w_k(x) \qquad \forall\, x \in \Omega\,.
\tag{5.4}
$$

Now, since $\alpha'(r)$ grows like $|r|^{2p}$ by (H1), we conclude that $\alpha'(w_k) \to \alpha'(w)$ and $\alpha(w_k) \to \alpha(w)$ in $L^q(\Omega)$ for all $1 \le q < +\infty$. Therefore, $\nabla\alpha(w_k) = \alpha'(w_k)\nabla w_k \to \alpha'(w)\nabla w$ in $L^\rho(\Omega)$ for all $\rho \in [1,2)$ and (5.3) follows. $\qquad\square$

Proof of Proposition 3.6. Thanks to Theorem 2, the set $\mathcal{S}$ complies with the existence axiom **(P1)** in Definition 2.4. The translation property **(P2)** is immediate to check. Concerning the concatenation axiom, let χ_1 and χ_2 be two solutions to Problem 2 on $(0,+\infty)$, satisfying $\chi_1(\tau) = \chi_2(0)$ for some $\tau \ge 0$, and let the functions w_1 and w_2 be such that, for $i = 1,2$, the pairs (χ_i, w_i) satisfy equations (3.18)–(3.19), with regularities (3.20) and (3.21). Then, one easily sees that the concatenations (cf. (2.23)) $\tilde{\chi}$ and $\tilde{w}$ of χ_1, χ_2 and w_1, w_2, respectively, satisfy equations (3.18)–(3.19), and still enjoy regularities (3.20) and (3.21), respectively (the fact that $\chi_1(\tau) = \chi_2(0)$ is crucial for the time-regularity of $\tilde{\chi}$).

To prove that all solutions $\chi \in \mathcal{S}$ are continuous w.r.t. the phase space topology (3.29), let us fix $\{t_n\}, t_0$ in $[0,+\infty)$, and show that

$$
t_n \to t_0 \ \Rightarrow\ \left(\|\chi(t_n) - \chi(t_0)\|_V + \left\| \widehat{\phi}(\chi(t_n)) - \widehat{\phi}(\chi(t_0)) \right\|_{L^1(\Omega)} \right) \to 0 \quad \text{as } n \to +\infty\,.
\tag{5.5}
$$

Indeed, thanks to regularity (3.20), for all $T > 0$, the function $\chi : [0, T] \to V$ is continuous w.r.t. the weak V-topology, hence

$$\chi(t_n) \rightharpoonup \chi(t_0) \qquad \text{in } V. \tag{5.6}$$

Therefore, by Lemma 5.1, we have

$$\liminf_{n \to +\infty} \mathcal{E}(\chi(t_n)) \geq \mathcal{E}(\chi(t_0)).$$

Combining this inequality with the continuity of the map $t \in [0, T] \mapsto \mathcal{E}(\chi(t))$, one concludes that

$$\lim_{n \to +\infty} \int_\Omega |\nabla \chi(t_n)|^2 = \int_\Omega |\nabla \chi(t_0)|^2, \tag{5.7a}$$

$$\lim_{n \to +\infty} \int_\Omega \widehat{\phi}(\chi(t_n)) = \int_\Omega \widehat{\phi}(\chi(t_0)). \tag{5.7b}$$

Clearly, (5.6), combined with (5.7a), yields that $\chi(t_n) \to \chi(t_0)$ in V. In order to prove the additional convergence

$$\|\widehat{\phi}(\chi(t_n)) - \widehat{\phi}(\chi(t_0))\|_{L^1(\Omega)} \to 0 \quad \text{as } n \to +\infty, \tag{5.8}$$

we note that (5.5) implies, in particular, that

$$\widehat{\phi}(\chi(\cdot, t_n)) \to \widehat{\phi}(\chi(\cdot, t_0)) \qquad \text{a.e. in } \Omega. \tag{5.9}$$

In view of [17, Lemma 4.2], (5.9), combined with (5.7b) and the fact that $\widehat{\phi}$ takes non-negative values, yields (5.8).

The energy identity (3.31) follows by multiplying (3.18) by w (note that the latter is an admissible test function, thanks to (3.21)), (3.19) by χ_t, adding the resulting relations, taking into account the chain rule (5.1) and formula (5.3), and integrating in time.

It remains to prove the upper-semicontinuity with respect to the initial data. To this aim, we will exploit (3.31). Thus, let us fix a sequence of solutions $\{\chi_n\} \subset \mathcal{S}$ and $\chi_0 \in X$, with

$$\mathrm{d}_X(\chi_n(0), \chi_0) \to 0 \text{ as } n \to +\infty, \text{ so that, in particular, } \mathcal{E}(\chi_n(0)) \to \mathcal{E}(\chi_0). \tag{5.10}$$

Identity (3.31) yields that there exists a constant $C > 0$ such that, for all $n \in \mathbb{N}$,

$$\delta \int_0^t \int_\Omega |\partial_t \chi_n|^2 + \int_0^t \int_\Omega \alpha'(w_n)|\nabla w_n|^2 + \mathcal{E}(\chi_n(t)) = \mathcal{E}(\chi_n(0)) \leq C \text{ for all } t \geq 0. \tag{5.11}$$

Arguing as in Section 4.1, we obtain estimates (4.2), (4.3), (4.13), (4.17), (4.26), and (4.28) for the sequence $\{(\chi_n, w_n)\}$, on every interval $(0, T)$. Therefore, with a diagonalization procedure, we find a subsequence $\{(\chi_{n_k}, w_{n_k})\}$ and functions $(\chi, w) : (0, +\infty) \to X \times V$ for which (4.40)–(4.42), (4.48), and (4.57) hold on every interval $(0, T)$, for all $T > 0$. Using all the aforementioned relations, we have $\chi(0) = \chi_0$ and, arguing as in Section 4.3, we conclude that $\chi \in \mathcal{S}$. In order to prove that

$$\text{for all } t \geq 0, \quad \left(\|\chi_{n_k}(t) - \chi(t)\|_V + \left\| \widehat{\phi}(\chi_{n_k}(t)) - \widehat{\phi}(\chi(t)) \right\|_{L^1(\Omega)} \right) \to 0 \quad \text{as } k \to +\infty, \tag{5.12}$$

we first obtain some *enhanced* convergence for the sequence $\{w_{n_k}\}$. To this aim, we note that, for every $T > 0$, there holds

$$\mathsf{c}_\alpha \limsup_{k \to +\infty} \int_0^T \int_\Omega |w_{n_k}|^{2p+2} \leq \limsup_{k \to +\infty} \int_0^T \int_\Omega \alpha(w_{n_k})w_{n_k} - \liminf_{k \to +\infty} \int_0^T \int_\Omega \Psi(w_{n_k})$$

$$\leq \int_0^T \int_\Omega \alpha(w)w - \int_0^T \int_\Omega \Psi(w) = \mathsf{c}_\alpha \int_0^T \int_\Omega |w|^{2p+2}.$$

Indeed, the first inequality follows from (H9), the second one from the second convergence in (4.57), and from (4.42), together with the convexity of Ψ (thanks to Ioffe's Theorem [12]), and the third one from (H9) again. Taking into account the fact that

$$\liminf_{k \to +\infty} \int_0^T \int_\Omega |w_{n_k}|^{2p+2} \geq \int_0^T \int_\Omega |w|^{2p+2},$$

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due to (4.42), we have

$$w_{n_k} \to w \qquad \text{in } L^{2p+2}(0,T;L^{2p+2}(\Omega)) \quad \text{for all } T > 0$$

and, thus, finally,

$$w_{n_k} \to w \qquad \text{in measure in } \Omega \times (0,T) \text{ for all } T > 0. \tag{5.13}$$

As a consequence, for all $t \geq 0$,

$$\liminf_{k \to +\infty} \int_0^t \int_\Omega \alpha'(w_{n_k}) |\nabla w_{n_k}|^2 \geq \int_0^t \int_\Omega \alpha'(w) |\nabla w|^2 \,, \tag{5.14}$$

thanks to the convergence in measure (5.13), the weak convergence (4.42) for $\{\nabla w_{n_k}\}$ in $L^2(0,T;H)$ for all $T > 0$, and again Ioffe's Theorem 2.2. Hence, passing to the limit in the energy identity (5.11) (written for the functions (χ_{n_k}, w_{n_k})), we infer, for all $t \geq 0$,

$$
\begin{aligned}
\delta \int_0^t &\int_\Omega |\partial_t \chi|^2 + \int_0^t \int_\Omega \alpha'(w) |\nabla w|^2 + \mathcal{E}(\chi(t)) \\
&\leq \liminf_{k \to +\infty} \left(\delta \int_0^t \int_\Omega |\partial_t \chi_{n_k}|^2 + \int_0^t \int_\Omega \alpha'(w_{n_k}) |\nabla w_{n_k}|^2 + \mathcal{E}(\chi_{n_k}(t)) \right) \\
&\leq \limsup_{k \to +\infty} \left(\delta \int_0^t \int_\Omega |\partial_t \chi_{n_k}|^2 + \int_0^t \int_\Omega \alpha'(w_{n_k}) |\nabla w_{n_k}|^2 + \mathcal{E}(\chi_{n_k}(t)) \right) \\
&= \lim_{k \to +\infty} \mathcal{E}(\chi_{n_k}(0)) = \mathcal{E}(\chi_0) = \delta \int_0^t \int_\Omega |\partial_t \chi|^2 + \int_0^t \int_\Omega \alpha'(w) |\nabla w|^2 + \mathcal{E}(\chi(t)) \,,
\end{aligned}
\tag{5.15}
$$

where the first inequality follows from (4.40)–(4.41), (5.14), and the fact that $\mathcal{E}$ is lower-semicontinuous w.r.t. the H-topology, the third one from (5.11), the fourth one from (5.10), and the last equality from the *energy identity* (3.31) satisfied by all solutions in $\mathcal{S}$. With an elementary argument, we deduce from (5.15) that, for all $t > 0$,

$$\int_0^t \int_\Omega |\partial_t \chi_{n_k}|^2 \to \int_0^t \int_\Omega |\partial_t \chi|^2, \quad \text{whence} \quad \chi_{n_k} \to \chi \text{ in } H^1(0,t;H),$$

as well as

$$\mathcal{E}(\chi_{n_k}(t)) \to \mathcal{E}(\chi(t)).$$

Arguing in the same way as throughout (5.6)–(5.9) and again invoking [17, Lemma 4.2], we obtain (5.12). This concludes the proof. $\qquad\square$

5.2 Proof of Theorem 3

Eventual boundedness. In order to check that $\mathcal{S}$ is eventually bounded, we fix a ball $B(0,R)$ centered at 0 of radius R in X, some initial datum $\chi_0 \in B_X(0,R)$, namely satisfying (recall that we can assume that $\widehat{\phi}$ is a positive function)

$$\|\chi_0\|_V + \int_\Omega \widehat{\phi}(\chi_0) \leq R, \tag{5.16}$$

and consider a generic trajectory $\chi \in \mathcal{S}$ starting from χ_0. Recalling the energy identity (3.31), we find, for all $t \geq 0$,

$$\int_\Omega \widehat{\phi}(\chi(t)) \leq \mathcal{E}(\chi(t)) \leq \mathcal{E}(\chi_0) \leq R, \qquad \int_\Omega |\nabla \chi(t)|^2 \leq 2R. \tag{5.17}$$

Now, taking into account the fact that $m(\chi(t)) = m(\chi_0)$ for all $t \geq 0$ (cf. (4.4)), we deduce from (5.17) a bound for $\|\chi\|_{L^\infty(0,+\infty;V)}$. Hence, there exists $R' > 0$ such that $d_X(\chi(t),0) \leq R'$ for all $t \geq 0$. Since χ_0 is arbitrary, we conclude that the evolution of the ball $B_X(0,R)$ is contained in the ball $B_X(0,R')$.

Compactness. In order to verify that $\mathcal{S}$ is compact, we consider a sequence $\{\chi_n\} \subset \mathcal{S}$ such that $\{\chi_n(0)\}$ is bounded in X. We write the energy identity (5.11) and, as in the proof of Proposition 3.6, deduce that there exist a subsequence $\{(\chi_{n_k}, w_{n_k})\}$ and functions $(\chi, w) : (0, +\infty) \to X \times V$ for which convergences (4.40)–(4.42), (4.48), and (4.57) hold on every interval $(0, T)$ for all $T > 0$. However, we cannot prove that

$$\left(\|\chi_{n_k}(t) - \chi(t)\|_V + \left\| \widehat{\phi}(\chi_{n_k}(t)) - \widehat{\phi}(\chi(t)) \right\|_{L^1(\Omega)} \right) \to 0 \quad \text{for all } t > 0, \tag{5.18}$$

arguing in the same way as throughout (5.12)–(5.15), for, in this case, we do not have the convergence of the initial energies $\mathcal{E}(\chi_{n_k}(0))$ at our disposal. Then, we rely on the following procedure (see also [20, 22] for the use of an analogous argument).

First, we apply Helly's compactness principle (with respect to the pointwise convergence) for monotone functions to the functions $t \mapsto \mathcal{E}(\chi_{n_k}(t))$, which are non-increasing in view of the energy identity (5.11). Thus, up to a (not relabeled) subsequence, there exists a non-increasing function $\mathscr{E} : [0, +\infty) \to \mathbb{R}$ such that

$$\mathscr{E}(t) := \lim_{k \to +\infty} \mathcal{E}(\chi_{n_k}(t)) \qquad \text{for all } t \geq 0. \tag{5.19}$$

By the lower-semicontinuity of $\mathcal{E}$ (w.r.t. the H-topology), we find

$$\mathcal{E}(\chi(t)) \leq \mathscr{E}(t) \qquad \text{for all } t \geq 0. \tag{5.20}$$

On the other hand, (4.40) ensures that, up to a further extraction, for almost all $s \in (0, t)$,

$$\chi_{n_k}(s) \to \chi(s) \text{ in } H^{2-\varepsilon}(\Omega) \text{ for all } \varepsilon > 0, \text{ whence } \chi_{n_k}(s) \to \chi(s) \text{ in } H^1(\Omega) \cap L^\infty(\Omega). \tag{5.21}$$

Thus, in particular,

$$\widehat{\phi}(\chi_{n_k}(\cdot, s)) \to \widehat{\phi}(\chi(\cdot, s)) \qquad \text{a.e. in } \Omega. \tag{5.22}$$

Moreover, for every $\mathcal{O} \subset \Omega$, there holds

$$\int_{\mathcal{O}} |\widehat{\phi}(\chi_{n_k}(s))| \leq |\mathcal{O}||\widehat{\phi}(0)| + \frac{C_{\phi,2}}{2} \int_{\mathcal{O}} |\chi_{n_k}(s)|^2 + \int_{\mathcal{O}} |\phi(\chi_{n_k}(s))||\chi_{n_k}(s)|$$
$$\leq C \left(|\mathcal{O}| + \int_{\mathcal{O}} |\phi(\chi_{n_k}(s))| \right), \tag{5.23}$$

where the first inequality follows from (3.17) and the second one from (5.21). Notice that the right-hand side of (5.23) tends to zero as $|\mathcal{O}| \to 0$, since the sequence $\{\phi(\chi_{n_k}(s))\}$ is uniformly integrable in $L^1(\Omega)$ thanks to (4.48). Hence, (5.23) yields that $\{\widehat{\phi}(\chi_{n_k}(s))\}$ is itself uniformly integrable in $L^1(\Omega)$. Combining this with (5.22), in view of Theorem 2.1 we conclude that $\widehat{\phi}(\chi_{n_k}(s)) \to \widehat{\phi}(\chi(s))$ in $L^1(\Omega)$. Finally, we have shown that there exists a negligible set $\mathcal{N} \subset (0, +\infty)$ such that

$$\mathscr{E}(s) = \lim_{k \to +\infty} \mathcal{E}(\chi_{n_k}(s)) = \mathcal{E}(\chi(s)) \qquad \text{for a.a. } s \in (0, +\infty) \setminus \mathcal{N}. \tag{5.24}$$

We are now in a position to carry out the argument for (5.18) (which bypasses the lack of convergence of the initial data in the phase space (3.29)), using the fact that the energy identity (5.11) holds for all $t > 0$. Indeed, for every fixed $t > 0$ and for all $s \in (0, t) \setminus \mathcal{N}$, we can pass to the limit in the energy identity (5.11), written for the sequence (χ_{n_k}, w_{n_k}) on the interval (s, t). Note indeed that convergences (4.40)–(4.42) and (4.48) for (χ_{n_k}, w_{n_k}) hold on (s, t). Proceeding as above, we then deduce once more that

$$\lim_{k \to +\infty} \int_s^t \int_\Omega \alpha(w_{n_k}) w_{n_k} = \int_s^t \int_\Omega \alpha(w) w,$$

whence $w_{n_k} \to w$ in $L^{2p+2}(s, t; L^{2p+2}(\Omega))$. Therefore, repeating the very same passages as in (5.15) and relying on (5.24), we find

$$\delta \int_s^t \int_\Omega |\partial_t \chi|^2 + \int_s^t \int_\Omega \alpha'(w)|\nabla w|^2 + \mathcal{E}(\chi(t))$$
$$= \lim_{k \to +\infty} \left(\delta \int_s^t \int_\Omega |\partial_t \chi_{n_k}|^2 + \int_s^t \int_\Omega \alpha'(w_{n_k})|\nabla w_{n_k}|^2 + \mathcal{E}(\chi_{n_k}(t)) \right),$$

which gives

$$\mathscr{E}(t) = \lim_{k \to +\infty} \mathcal{E}(\chi_{n_k}(t)) = \mathcal{E}(\chi(t)) \quad \text{for all } t > 0,$$

and, finally, (5.18).

Lyapunov function and rest points. We now verify that $\mathcal{E}$ acts as a Lyapunov functional for $\mathcal{S}$. Actually, $\mathcal{E}$ clearly is continuous on X and decreasing along all solutions, thanks to the energy identity (3.31). Furthermore, assume that, along some $\chi \in \mathcal{S}$, the map $t \in [0, +\infty) \mapsto \mathcal{E}(\chi(t))$ is constant. Then, in view of (3.31), we find $\nabla w \equiv 0$ and $\chi_t \equiv 0$ a.e. in $(0, +\infty)$, so that $\chi(t) \equiv \chi(0)$ for all $t \in [0, +\infty)$. Analogously, we immediately find that $\bar{\chi} \in X$ is a rest point for $\mathcal{S}$ if and only if it satisfies the stationary system

$$A(\alpha(\bar{w})) = 0 \quad \text{a.e. in } \Omega, \tag{5.25a}$$
$$A\bar{\chi} + \phi(\bar{\chi}) = \bar{w} \quad \text{a.e. in } \Omega. \tag{5.25b}$$

Conclusion of the proof. We apply Theorem 2.3 and Remark 2.5 with the choice $\mathcal{D} := \mathcal{D}_{m_0}$ for some $m_0 > 0$ (cf. (3.32)). Thanks to (4.4) (recall the second a priori estimate in Section 4.1), for all $\chi_0 \in \mathcal{D}_{m_0}$, every solution starting from the initial datum χ_0 remains in $\mathcal{D}_{m_0}$, so that the first condition in (2.28) is satisfied. To check the second one, we fix some $\bar{\chi} \in Z(\mathcal{S})$ with $|m(\bar{\chi})| \leq m_0$. It follows from (5.25a) and (H1) that $\nabla \bar{w} \equiv 0$, so that $\bar{w}$ is constant in Ω. Hence, we test (5.25b) by $\bar{\chi} - m(\bar{\chi})$. Since $\bar{w} = m(\bar{w})$, we infer that

$$\|\nabla \bar{\chi}\|_H^2 + \int_\Omega \phi(\bar{\chi})(\bar{\chi} - m(\bar{\chi})) \leq 0. \tag{5.26}$$

On the other hand, (H10) ensures that estimate (3.5) holds, so that there exist constants $\mathcal{K}_{m_0}, \mathcal{K}^1_{m_0} > 0$, only depending on m_0, such that

$$\int_\Omega |\phi(\bar{\chi})| \leq \mathcal{K}_{m_0} \int_\Omega \phi(\bar{\chi})(\bar{\chi} - m(\bar{\chi})) + \mathcal{K}^1_{m_0}. \tag{5.27}$$

Collecting (5.26) and (5.27), we deduce that

$$\|\nabla \bar{\chi}\|_H^2 + \frac{1}{\mathcal{K}_{m_0}} \int_\Omega |\phi(\bar{\chi})| \leq \frac{\mathcal{K}^1_{m_0}}{\mathcal{K}_{m_0}},$$

whence, in particular,

$$|m(\bar{w})| = |m(\phi(\bar{\chi}))| \leq \frac{\mathcal{K}^1_{m_0}}{|\Omega|}.$$

Taking into account the fact that $\nabla \bar{w} = 0$ (so that $\bar{w}$ is a constant) and that $|m(\bar{\chi})| \leq m_0$, we conclude that

$$\exists \mathcal{K}^2_{m_0} > 0 : \quad \forall \bar{\chi} \in Z(\mathcal{S}) \cap \mathcal{D}_{m_0} \quad \|\bar{\chi}\|_V + |\bar{w}| \leq \mathcal{K}^2_{m_0}. \tag{5.28}$$

Thus, a comparison in (5.25b) and the standard elliptic regularity estimate (cf. also the calculations developed throughout (4.14)–(4.15)), yield

$$\exists \mathcal{K}^3_{m_0} > 0 : \quad \forall \bar{\chi} \in Z(\mathcal{S}) \cap \mathcal{D}_{m_0} \quad \|\phi(\bar{\chi})\|_{L^6(\Omega)} + \|\bar{\chi}\|_{W^{2,6}(\Omega)} \leq \mathcal{K}^3_{m_0}, \tag{5.29}$$

whence, in particular, an estimate for $\bar{\chi}$ in $L^\infty(\Omega)$. Then, using (3.17), we readily infer that

$$\exists \mathcal{K}^4_{m_0} > 0 : \quad \forall \bar{\chi} \in Z(\mathcal{S}) \cap \mathcal{D}_{m_0} \quad \|\widehat{\phi}(\bar{\chi})\|_{L^6(\Omega)} \leq \mathcal{K}^4_{m_0}. \tag{5.30}$$

Finally, (5.28) and (5.30) yield that $Z(\mathcal{S}) \cap \mathcal{D}_{m_0}$ is bounded in the phase space X, and the existence of the global attractor follows from Theorem 2.3.

In fact, with the same calculations as in the above lines, joint with a boot-strap argument, one easily proves that

$$\forall p \in [1, +\infty) : \quad \exists C_p > 0 \quad \bar{\chi} \in Z(\mathcal{S}) \cap \mathcal{D}_{m_0} \quad \|\bar{\chi}\|_{W^{2,p}(\Omega)} + \|\widehat{\phi}(\bar{\chi})\|_{L^p(\Omega)} \leq C_p. \tag{5.31}$$

Then, estimate (3.34) is a straightforward consequence of (2.27) and (5.31). $\square$

6   Proof of Theorems 3.1 and 3.2

Proof of Theorem 3.1. Within this proof, we denote by c_δ a positive constant depending on $\delta > 0$ and on quantities (3.26). Referring to the notation of the statement of Theorem 3.1, let us set $\underline{\chi}_0 := \chi_0^1 - \chi_0^2$, $\underline{\chi} := \chi_1 - \chi_2$, and $\underline{w} := w_1 - w_2$. The pair $(\underline{\chi}, \underline{w})$ obviously satisfies

$$\underline{\chi}_t + A(\alpha(w_1)) - A(\alpha(w_2)) = 0 \quad \text{a.e. in } \Omega \times (0, T), \tag{6.1}$$

$$\delta\underline{\chi}_t + A\underline{\chi} + \chi_1^3 - \chi_2^3 - \underline{\chi} = \underline{w} \quad \text{a.e. in } \Omega \times (0, T). \tag{6.2}$$

Following the proof of [18, Prop. 2.1], we test (6.1) by $\mathcal{N}(\underline{w} - m(\underline{w}))$, (6.2) by $\mathcal{N}(\underline{\chi}_t) + \underline{\chi}$, add the resulting equations, and integrate over $(0, t)$, $t \in (0, T)$. We refer to the proof of [18, Prop. 2.1] for all the detailed computations, leading to (cf. [18, (3.51)])

$$\int_0^t \|\underline{w}\|_H^2 + \delta \int_0^t \|\mathcal{N}(\underline{\chi}_t)\|_V^2 + \delta\|\underline{\chi}(t)\|_H^2 + \int_0^t \|\nabla\underline{\chi}\|_H^2 \le C\left(\|\underline{\chi}_0\|_H^2 + \int_0^t \|\underline{\chi}\|_H^2\right). \tag{6.3}$$

An easy application of Gronwall's lemma to the function $t \mapsto \|\underline{\chi}(t)\|_H^2$ entails

$$\|\underline{\chi}\|_{C^0([0,t];H) \cap L^2(0,t;V)} + \|\underline{\chi}_t\|_{L^2(0,t;V')} + \|\underline{w}\|_{L^2(0,t;H)} \le c_\delta\|\underline{\chi}_0\|_H. \tag{6.4}$$

Furthermore, exploiting (H8) and the above (6.4), it follows from the Hölder inequality that

$$\begin{aligned}
&\|\phi(\chi_1) - \phi(\chi_2)\|_{L^2(0,t;H)}^2 \\
&\le C \int_0^t \int_\Omega |\underline{\chi}|^2 \left(\chi_1^2 + \chi_2^2 + 1\right)^2 \\
&\le C \int_0^t \left(\|\chi_1\|_{L^6(\Omega)}^4 + \|\chi_2\|_{L^6(\Omega)}^4\right) \|\underline{\chi}\|_{L^6(\Omega)}^2 + C \int_0^t \int_\Omega |\underline{\chi}|^2 \\
&\le C \left(\|\chi_1\|_{L^\infty(0,T;L^6(\Omega))}^4 + \|\chi_2\|_{L^\infty(0,T;L^6(\Omega))}^4 + 1\right) \|\underline{\chi}\|_{L^2(0,t;V)}^2 \le c_\delta\|\underline{\chi}_0\|_H^2.
\end{aligned} \tag{6.5}$$

Next, we test (6.2) by $\underline{\chi}_t$ and integrate in time to obtain

$$\frac{\delta}{2} \int_0^t \|\underline{\chi}_t\|_H^2 + \frac{1}{2}\|\nabla(\underline{\chi}(t))\|_H^2 \le \frac{1}{2}\|\nabla\underline{\chi}_0\|_H^2 + c_\delta\left(\int_0^t \|\underline{w}\|_H^2 + \int_0^t \|\phi(\chi_1) - \phi(\chi_2)\|_H^2 + \int_0^t \|\underline{\chi}\|_H^2\right). \tag{6.6}$$

In view of (6.4)–(6.6), we readily infer the continuous dependence estimate (3.27) for $\underline{\chi}$ in $C^0([0, t]; V) \cap H^1(0, t; H)$. Then, the estimate for $\|\underline{\chi}\|_{L^2(0,t;Z)}$ follows from (6.4)–(6.6) by a comparison argument. □

Proof of Theorem 3.2. Referring to the notation of the proof of Theorem 3.1, we again test (6.1) by $\mathcal{N}(\underline{w} - m(\underline{w}))$, (6.2) by $\mathcal{N}(\underline{\chi}_t) + \underline{\chi}$, add the resulting equations, and integrate over $(0, t)$, $t \in (0, T)$. Developing the same calculations as in the above lines, we note that the chain of inequalities (6.5) is now trivial, since under the present assumptions the functions χ_1 and χ_2 are estimated in $L^\infty(0, T; Z)$ (see Proposition 3.3). On the other hand, the following term:

$$I := \int_0^t \int_\Omega m(\underline{w})\left(\alpha(w_1) - \alpha(w_2)\right),$$

which was easily estimated in the proof of Theorem 3.1, now needs to be carefully handled because of the (at most) quadratic controlled growth of α'. Indeed, observe that

$$
\begin{aligned}
|I| &\leq \int_0^t \|m(\underline{w})\|_{L^\infty(\Omega)} \|\alpha(w_1) - \alpha(w_2)\|_{L^1(\Omega)} \\
&\leq C \int_0^t \left(\|m(\underline{w})\|_{L^1(\Omega)} \int_\Omega (1 + |w_1|^{2p} + |w_2|^{2p})\underline{w} \right) \\
&\leq C \int_0^t \left(\|\phi(\chi_1) - \phi(\chi_2)\|_{L^1(\Omega)} (1 + \|w_1\|^{2p}_{L^\infty(\Omega)} + \|w_2\|^{2p}_{L^\infty(\Omega)})\|\underline{w}\|_{L^1(\Omega)} \right) \\
&\leq C \int_0^t \left(\|\underline{\chi}\|_{L^1(\Omega)} (1 + \|w_1\|^{2p}_{L^\infty(\Omega)} + \|w_2\|^{2p}_{L^\infty(\Omega)})\|\underline{w}\|_{L^1(\Omega)} \right) \\
&\leq \varrho \int_0^t \|\underline{w}\|_H^2 + C_\varrho \int_0^t (1 + \|w_1\|^{4p}_{L^\infty(\Omega)} + \|w_2\|^{4p}_{L^\infty(\Omega)})\|\underline{\chi}\|_H^2
\end{aligned}
$$

for some $\varrho \in (0,1)$ and $C_\varrho > 0$. This modification gives, in place of (6.3),

$$
\begin{aligned}
(1 - \varrho) \int_0^t \|\underline{w}\|_H^2 &+ \delta \int_0^t \|\mathcal{N}(\underline{\chi}_t)\|_V^2 + \delta\|\underline{\chi}(t)\|_H^2 + \int_0^t \|\nabla\underline{\chi}\|_H^2 \\
&\leq C \left(\|\underline{\chi}_0\|_H^2 + \int_0^t (1 + \|w_1\|^{4p}_{L^\infty(\Omega)} + \|w_2\|^{4p}_{L^\infty(\Omega)})\|\underline{\chi}\|_H^2 \right).
\end{aligned}
$$

Thus, recalling (3.21), we can use Gronwall's lemma to deduce (6.4). Estimate (6.6) can be obtained by arguing as in the proof of Theorem 3.1, hence the result. $\qquad\square$

A   Appendix

We propose the following approximate system for *both* Problem 1 and Problem 2:

$$
\begin{aligned}
\chi_t + A(\alpha_M(w)) &= 0 &&\text{a.e. in } \Omega \times (0,T)\,, &&\text{(A.1)} \\
\delta\chi_t + A\chi + \phi_\mu(\chi) &= w &&\text{a.e. in } \Omega \times (0,T)\,, &&\text{(A.2)}
\end{aligned}
$$

depending on the parameters δ, M, $\mu > 0$, where

$$
\alpha_M(r) = \begin{cases} \alpha(-M) + C_1(r - M) & \text{if } r < -M, \\ \alpha(r) & \text{if } |r| \leq M, \\ \alpha(M) + C_1(r - M) & \text{if } r > M, \end{cases} \tag{A.3}
$$

C_1 being the same constant as in (H1), and

$$
\phi_\mu(r) = \begin{cases} \phi(r) & \text{if } |\phi(r)| \leq \frac{1}{\mu}, \\ \frac{1}{\mu}\operatorname{sign}(r) & \text{otherwise.} \end{cases} \tag{A.4}
$$

It is immediate to check that, for any choice of the approximation parameters M and μ, the functions α_M and ϕ_μ are Lipschitz continuous on $\mathbb{R}$ and that

$$
\begin{aligned}
\alpha_M &\to \alpha &&\text{uniformly on compact subsets of } \mathbb{R} \text{ as } M \nearrow +\infty, \\
\phi_\mu &\to \phi &&\text{uniformly on compact subsets of } \operatorname{dom}(\phi) \text{ as } \mu \searrow 0.
\end{aligned} \tag{A.5}
$$

Of course, the Lipschitz constants of α_M and ϕ_μ explode as $M \nearrow +\infty$ and $\mu \searrow 0$, respectively. Let us also point out that, by construction,

$$
\alpha_M'(r) \geq C_1 > 0 \qquad \text{for all } r \in \mathbb{R},\ M > 0, \tag{A.6}
$$

which yields that the inverse $\rho_M : \mathbb{R} \to \mathbb{R}$ of α_M is Lipschitz continuous, with

$$|\rho_M(x) - \rho_M(y)| \leq \frac{1}{C_1}|x - y| \quad \text{for all } x, y \in \mathbb{R}, \, M > 0. \tag{A.7}$$

What is more, relying on convergence (A.5) of ϕ_μ to ϕ, one can also check that, for $\mu > 0$ sufficiently small (say $0 < \mu \leq \mu_*$), (3.5) and (H3) hold on this approximate level as well, i.e.,

$$\forall\, m \in \mathrm{dom}(\phi) = (a,b) \;\; \exists\, C_m, \, C'_m > 0 : \;\; \forall\, 0 < \mu \leq \mu_* \quad \forall\, r \in (a-m, b-m)$$
$$|\phi_\mu(r+m)| \leq C_m \phi_\mu(r+m)r + C'_m\,, \tag{A.8}$$

as well as

$$\exists\, C > 0 : \;\; \forall\, 0 < \mu \leq \mu_* \qquad \forall\, r \in (a,b) \quad |\phi_\mu(r)|^\sigma \leq C\left(\widehat{\phi_\mu}(r) + 1\right)\,, \tag{A.9}$$

with $\sigma \in (0,1)$ as in (H3), in particular, complying with the compatibility condition (H4).

It was proved in [18, Thm. 2.1] that, for every $\delta, M, \mu > 0$, there exists a unique pair (χ, w), with

$$\chi \in L^2(0,T;Z) \cap L^\infty(0,T;V) \cap H^1(0,T;H),$$
$$w \in L^2(0,T;V), \tag{A.10}$$

solving the Cauchy problem for system (A.1)–(A.2), supplemented with some initial datum $\chi^0 \in V$.

Problem $\mathbf{P}_{\delta,\mu}$. In what follows, we approximate the initial datum $\chi_0 \in V$ in (3.12) by a sequence

$$\{\chi_{0,\mu}\} \subset H^4(\Omega) \quad \text{with} \quad \chi_{0,\mu} \rightharpoonup \chi_0 \;\; \text{in } V \;\; \text{and} \;\; \sup_{\mu > 0}\|\widehat{\phi}_\mu(\chi_{0,\mu})\|_{L^1(\Omega)} < +\infty \tag{A.11}$$

(for example, we can construct $\{\chi_{0,\mu}\}$ by applying (twice) the elliptic regularization procedure developed in the proof of [4, Prop. 2.6]).

For every $\delta, M, \mu > 0$, we call $\mathbf{P}_{\delta,M,\mu}$ the initial and boundary value problem obtained by supplementing the PDE system (A.1)–(A.2) with the initial condition

$$\chi(0) = \chi_{0,\mu} \quad \text{in } H^4(\Omega). \tag{A.12}$$

In the following Section A.1, we will prove some further regularity of the approximate solutions. In this way, we will justify, on the level of the approximate Problem $\mathbf{P}_{\delta,M,\mu}$, the estimates formally performed in Section 4.1. Hence, in Section A.2, we will develop the rigorous proof of Theorem 1 by relying on the aforementioned estimates and by passing to the limit in Problem $\mathbf{P}_{\delta,M,\mu}$, first as $\delta \searrow 0$ for $M, \mu > 0$ fixed, then as $M \nearrow +\infty$ for $\mu > 0$ fixed, and, finally, as $\mu \searrow 0$.

Furthermore, it would be possible to give a rigorous proof of Theorem 2 by passing to the limit in Problem $\mathbf{P}_{\delta,M,\mu}$ first as $M \nearrow +\infty$ for $\mu > 0$ fixed, and then as $\mu \searrow 0$. However, we are not going to enter into the details of the latter procedure, which follows the very same lines as the one for Theorem 1.

Notation A.1. In what follows, we denote by $C_{\delta,M,\mu}$ various constants (which can differ from occurrence to occurrence, even in the same line), depending on the parameters δ, M, and μ, and such that $C_{\delta,M,\mu} \nearrow +\infty$ as either $\delta \searrow 0$, or $M \nearrow +\infty$, or $\mu \searrow 0$. The symbols $C_{\delta,\mu}$, $C_{M,\mu}$, and C_μ have an analogous meaning.

A.1 Enhanced regularity estimates on the approximate problem

First estimate. We note that $w \in L^2(0,T;V)$ and that, since ϕ_μ is a Lipschitz continuous function, $\chi \in L^\infty(0,T;V)$ (cf. (A.10)) implies $\phi_\mu(\chi) \in L^\infty(0,T;V)$. Thus, by comparison in (A.2), we have $\delta\partial_t\chi + A\chi \in L^2(0,T;V)$. Hence, testing (A.2) by $A(\partial_t\chi)$ and using the fact that $\chi(0) = \chi_{0,\mu} \in Z$, we deduce the estimate

$$\|\nabla\partial_t\chi\|_{L^2(0,T;H)} + \|A\chi\|_{L^\infty(0,T;H)} \leq C_{\delta,\mu}, \tag{A.13}$$

whence

$$\chi \in L^\infty(0,T;Z) \cap H^1(0,T;V). \tag{A.14}$$

Second estimate. Since α_M is Lipschitz continuous and $w \in L^2(0,T;V)$, we have $\alpha_M(w) \in L^2(0,T;V)$. Estimate (A.13) and a comparison in (A.1) yield a bound for $A(\alpha_M(w))$ in $L^2(0,T;V)$, whence

$$\alpha_M(w) \in L^2(0,T;H^3(\Omega)) \subset L^2(0,T;W^{1,\infty}(\Omega)).$$

Recalling (A.7) and using the fact that $w = \rho_M(\alpha_M(w))$, we readily deduce the estimate

$$\|w\|_{L^2(0,T;W^{1,\infty}(\Omega))} \leq C_{\delta,M,\mu}. \tag{A.15}$$

Third estimate. Using a parabolic regularity argument in (A.2) and relying on regularity (A.11) for the approximate initial datum $\chi_{0,\mu}$, we deduce that

$$\|\partial_t \chi\|_{L^2(0,T;W^{1,3+\epsilon}(\Omega))} + \|A\chi\|_{L^2(0,T;W^{1,3+\epsilon}(\Omega))} \leq C_{\delta,M,\mu}, \tag{A.16}$$

where $\epsilon > 0$ is a suitable number. More precisely, since $\chi_{0,\mu} \in H^4(\Omega) \subset W^{3,6}(\Omega)$, the above formula holds for any $\epsilon \in (0,3]$ (cf. inequality (A.26) below for a justification). Thus, by interpolation, we obtain that $\nabla\chi$ belongs to $H^{1/2}(0,T;W^{1,3+\epsilon}(\Omega))$ and, recalling the continuous embedding $W^{1,3+\epsilon}(\Omega) \subset L^\infty(\Omega)$, we conclude that

$$\|\nabla\chi\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C_{\delta,M,\mu}. \tag{A.17}$$

Fourth estimate. Notice that, for almost all $t \in (0,T)$, the function $\nabla(|w(t)|^p w(t)) = (p+1)|w(t)|^p \nabla w(t)$ belongs to $L^2(\Omega)$, thanks to (A.10) and (A.15). Hence, for a.a. $t \in (0,T)$, we can test (A.2) by $|w(t)|^p w(t)$, which yields

$$
\begin{aligned}
\int_\Omega |w(t)|^{p+2} \\
= \int_\Omega \nabla\chi(t) \cdot \nabla(|w(t)|^p w(t)) + \int_\Omega \phi_\mu(\chi(t))|w(t)|^p w(t) + \delta \int_\Omega \partial_t \chi(t)|w(t)|^p w(t) \\
= (p+1)\int_\Omega |w(t)|^p \nabla\chi(t) \cdot \nabla w(t) + \int_\Omega \phi_\mu(\chi(t))|w(t)|^p w(t) \\
- \delta(p+1)\int_\Omega \alpha'_M(w(t))|w(t)|^p |\nabla w(t)|^2,
\end{aligned} \tag{A.18}
$$

the second equality following from equation (A.1). We estimate the second term on the right-hand side of the above equality by using the bound for $\phi_\mu(\chi)$ in $L^\infty(0,T;L^\infty(\Omega))$, due to (A.14) and the Lipschitz continuity of ϕ_μ. We deal with the first integral term as follows:

$$
\begin{aligned}
\left| \int_\Omega |w(t)|^p \nabla\chi(t) \cdot \nabla w(t) \right| &\leq \left\| |w(t)|^{p/2} \nabla w(t) \right\|_{L^2(\Omega)} \left\| |w(t)|^{p/2} \right\|_{L^2(\Omega)} \left\| \nabla\chi(t) \right\|_{L^\infty} \\
&\leq \varrho \int_\Omega |w(t)|^p |\nabla w(t)|^2 + C_{\delta,M,\mu} \int_\Omega |w(t)|^p
\end{aligned} \tag{A.19}
$$

for some suitable positive constant ϱ, where we have also used (A.17). Now, recalling (A.6), we estimate the last summand on the right-hand side of (A.18) by

$$-\delta(p+1)\int_\Omega \alpha'_M(w(t))|w(t)|^p |\nabla w(t)|^2 \leq -\delta(p+1)C_1 \int_\Omega |w(t)|^p |\nabla w(t)|^2,$$

and we move the above term to the left-hand side of (A.18). Then, we combine the latter inequality with (A.19), in which we choose $\varrho = \frac{\delta(p+1)C_1}{4}$. We thus obtain, for a.a. $t \in (0,T)$,

$$\int_\Omega |w(t)|^{p+2} + \frac{3}{4}\delta(p+1)C_1 \int_\Omega |w(t)|^p |\nabla w(t)|^2 \leq C_{\delta,M,\mu}\left(\int_\Omega |w(t)|^{p+1} + \int_\Omega |w(t)|^p \right). \tag{A.20}$$

Thus, we finally infer that

$$w \in L^\infty(0,T;L^p(\Omega)) \quad \text{for all } p \in [1,\infty), \tag{A.21}$$

whence, by the Lipschitz continuity of α_M,

$$\alpha_M(w) \in L^\infty(0,T;L^p(\Omega)) \quad \text{for all } p \in [1,+\infty). \tag{A.22}$$

A.2 Rigorous proof of Theorem 1

Within this section, for all $\delta,\ \mu > 0$, we will denote by $\{(\chi_{\delta,M,\mu}, w_{\delta,M,\mu})\}_{\delta,M,\mu}$ the family of solutions to Problem $\mathbf{P}_{\delta,M,\mu}$.

First step. For fixed $\mu, M > 0$, we pass to the limit in Problem $\mathbf{P}_{\delta,\mu}$ as $\delta \searrow 0$. We then perform the same calculations as in Section 4.1 (cf. (4.1)–(4.2), (4.6), (4.7)–(4.10)). Also relying on (A.8)–(A.9), we conclude that

$$
\begin{aligned}
\exists\, C > 0 : \quad \forall \delta,\, M,\, \mu > 0 \quad &\|\chi_{\delta,M,\mu}\|_{L^\infty(0,T;V)} + \|w_{\delta,M,\mu}\|_{L^2(0,T;V)} + \|\widehat{\phi}_\mu(\chi_{\delta,M,\mu})\|_{L^\infty(0,T;L^1(\Omega))} \\
&+ \delta^{1/2}\|\partial_t \chi_{\delta,M,\mu}\|_{L^2(0,T;L^2(\Omega))} + \|(\alpha'_M(w_{\delta,M,\mu}))^{1/2}\nabla w_{\delta,M,\mu}\|_{L^2(0,T;L^2(\Omega))} \leq C.
\end{aligned}
\tag{A.23}
$$

Recalling the definition of ϕ_μ and its Lipschitz continuity, we also have

$$
\exists\, C_\mu > 0 : \quad \forall \delta,\, M > 0 \quad \|\phi_\mu(\chi_{\delta,M,\mu})\|_{L^\infty(0,T;V)\cap L^\infty(0,T;L^\infty(\Omega))} \leq C_\mu.
\tag{A.24}
$$

In the same way, estimate (A.23) for $w_{\delta,M,\mu}$ and the Lipschitz continuity of α_M yield

$$
\exists\, C_M > 0 : \quad \forall \delta,\, \mu > 0 \quad \|\alpha_M(w_{\delta,M,\mu})\|_{L^2(0,T;V)} \leq C_M.
\tag{A.25}
$$

Next, a comparison in (3.19) and the maximal parabolic regularity result from [11] yield

$$
c(\delta)\int_0^T \|\partial_t \chi_{\delta,M,\mu}\|_{L^6(\Omega)}^2 + \int_0^T \|A\chi_{\delta,M,\mu}\|_{L^6(\Omega)}^2 \leq C\int_0^T \|\ell_{\delta,M,\mu}\|_{L^6(\Omega)}^2,
\tag{A.26}
$$

for some $c(\delta)$ such that $c(\delta) \to 0$ as $\delta \to 0$, where we have set

$$
\ell_{\delta,M,\mu} = w_{\delta,M,\mu} - \phi_\mu(\chi_{\delta,M,\mu}) - A\chi_{0,\mu}.
$$

In view of estimates (A.23) for $w_{\delta,M,\mu}$, (A.24) for $\phi_\mu(\chi_{\delta,M,\mu})$ in $L^\infty(0,T;V)$, and (A.11) for $\{\chi_{0,\mu}\}$, we conclude that

$$
\|\ell_{\delta,M,\mu}\|_{L^2(0,T;L^6(\Omega))} \leq C_\mu.
$$

Therefore, (A.26) gives

$$
\exists\, C_\mu > 0 : \quad \forall \delta,\, M > 0 \quad \|\chi_{\delta,M,\mu}\|_{L^2(0,T;W^{2,6}(\Omega))} \leq C_\mu.
\tag{A.27}
$$

On the other hand, estimate (A.25) and a comparison in (A.1) imply

$$
\exists\, C_M > 0 : \quad \forall \delta,\, \mu > 0 \quad \|\partial_t \chi_{\delta,M,\mu}\|_{L^2(0,T;V')} \leq C_M.
\tag{A.28}
$$

On behalf of the above estimates and arguing in the very same way as in Section 4.2, we see that, for every fixed $M > 0$ and $\mu > 0$, there exist a sequence $\delta_k \searrow 0$ (for notational simplicity, we do not highlight its dependence on the parameters M and μ) and functions $(\chi_{M,\mu}, w_{M,\mu}, \bar{\alpha}_{M,\mu})$ such that the sequence $\{(\chi_{\delta_k,M,\mu}, w_{\delta_k,M,\mu})\}$ converges to $(\chi_{M,\mu}, w_{M,\mu})$, as $k \to +\infty$, in the sense specified by (4.29)–(4.30), (4.32), as well as

$$
\begin{aligned}
\partial_t \chi_{\delta_k,M,\mu} &\rightharpoonup \partial_t \chi_{M,\mu} \quad \text{in } L^2(0,T;V'), \\
\delta_k^{1/2}\partial_t \chi_{\delta_k,M,\mu} &\rightharpoonup 0 \quad \text{in } L^2(0,T;H), \\
\alpha_M(w_{\delta_k,M,\mu}) &\rightharpoonup \bar{\alpha}_{M,\mu} \quad \text{in } L^2(0,T;V).
\end{aligned}
$$

Next, arguing similarly to the (formal) proof of Theorem 1, we conclude that

$$
\phi_\mu(\chi_{\delta_k,M,\mu}) \to \phi_\mu(\chi_{M,\mu}) \quad \text{in } L^2(0,T;H).
\tag{A.29}
$$

Finally, we use (A.29) in the very same way as in Section 4.2 to infer

$$
\bar{\alpha}_{M,\mu} = \alpha_M(w_{M,\mu})
$$

and

$$
\alpha_M(w_{\delta_k,M,\mu}) \to \alpha_M(w_{M,\mu}) \quad \text{in } L^2(0,T;H).
$$

Therefore, we conclude that the pair $(\chi_{M,\mu}, w_{M,\mu})$ is a solution to the PDE system

$$\chi_t + A(\alpha_M(w)) = 0 \qquad \text{a.e. in } \Omega \times (0,T), \tag{A.30}$$

$$A\chi + \phi_\mu(\chi) = w \qquad \text{a.e. in } \Omega \times (0,T), \tag{A.31}$$

supplemented with the initial condition (A.12).

Second step. We now take the limit $M \nearrow +\infty$ in (the Cauchy problem for) (A.30)–(A.31). Estimates (A.23) and (A.24) hold for the sequence of solutions $\{(\chi_{M,\mu}, w_{M,\mu})\}_M$ as well. Furthermore, using a lower-semicontinuity argument, we also deduce from (A.27) that

$$\|\chi_{M,\mu}\|_{L^2(0,T;W^{2,6}(\Omega))} \leq C_\mu \quad \text{for all } M > 0. \tag{A.32}$$

Now, we point out that (A.23) entails

$$\int_0^T \int_\Omega \alpha'_M(w_{M,\mu}) |\nabla w_{M,\mu}|^2 \leq C \quad \text{for all } M > 0. \tag{A.33}$$

Let us denote by $\mathcal{T}_M$ the truncation operator at level M and define

$$\tau_M := \mathcal{T}_M(w_{M,\mu}) := \begin{cases} -M & \text{if } w_{M,\mu} < -M, \\ w_{M,\mu} & \text{if } |w_{M,\mu}| \leq M, \\ M & \text{if } w_{M,\mu} > M, \end{cases} \qquad \text{a.e. } t \in \Omega \times (0,T) \tag{A.34}$$

(to simplify, we omit the index μ in the notation for τ_M). For later use, we also introduce for a.a. $t \in (0,T)$ the sets

$$\begin{cases} \mathcal{A}_M := \{(x,t) \in \Omega \times (0,T) \,:\, |w_{M,\mu}(x,t)| \leq M\}, \\ \mathcal{O}_M := \{(x,t) \in \Omega \times (0,T) \,:\, |w_{M,\mu}(x,t)| > M\}, \\ \mathcal{O}_M^t := \{x \in \Omega \,:\, (x,t) \in \mathcal{O}_M\}. \end{cases} \tag{A.35}$$

From (A.33), we also infer

$$\int_0^T \int_\Omega \alpha'(\tau_M) |\nabla \tau_M|^2 \leq C \quad \text{for all } M > 0,$$

whence, in view of (H1),

$$\||\tau_M|^p \nabla \tau_M\|_{L^2(0,T;H)} \leq C \quad \text{for all } M > 0. \tag{A.36}$$

Now, in order to reproduce estimates (4.23)–(4.24b) in the present approximate setting, we test (A.31) by $|\tau_M(t)|^p \tau_M(t)$ for a.e. $t \in (0,T)$. Clearly,

$$\int_\Omega w_{M,\mu}(t) |\tau_M(t)|^p \tau_M(t) \geq \int_\Omega |\tau_M(t)|^{p+2},$$

so that we have (cf. also (A.18))

$$\int_\Omega |\tau_M(t)|^{p+2} \leq (p+1) \int_\Omega |\tau_M(t)|^p \nabla \chi_M(t) \cdot \nabla \tau_M(t) + \int_\Omega \phi_\mu(\chi_{M,\mu}(t)) |\tau_M(t)|^p \tau_M(t)$$

$$\leq (p+1) \||\tau_M(t)|^p \nabla \tau_M(t)\|_H \|\nabla \chi_M(t)\|_H + C_\mu \int_\Omega |\tau_M(t)|^{p+1} \tag{A.37}$$

$$\leq C \||\tau_M(t)|^p \nabla \tau_M(t)\|_H + \frac{1}{2} \int_\Omega |\tau_M(t)|^{p+2} + C_\mu,$$

where the second inequality follows from estimate (A.24) and the last one from (A.23) and Young's inequality. Therefore, combining (A.36) and (A.37), we find an estimate for $\|\tau_M\|_{L^{p+2}(\Omega)}^{p+2}$ in $L^2(0,T)$ with some constant $C_\mu > 0$ which is independent of $M > 0$. On behalf of (A.34)–(A.35), from the latter bound, we infer (recall that $|\cdot|$ also denotes the Lebesgue measure)

$$C_\mu \geq \int_0^T \left(\int_{\mathcal{O}_M^t} |M|^{p+2} \, dx \right)^2 \, dt = M^{2p+4} \int_0^T |\mathcal{O}_M^t|^2 \, dt \geq \frac{M^{2p+4}}{T} |\mathcal{O}_M|^2 \quad \text{for all } M > 0, \tag{A.38}$$

29

where the last inequality is a direct consequence of Jensen's inequality. Next, we apply the nonlinear Poincaré inequality (2.15) to $|\tau_M|^p \tau_M$, thus obtaining (cf. (4.22))

$$\||\tau_M|^p \tau_M\|_V \le K \left(\|\nabla(|\tau_M|^p \tau_M)\|_H + |m(\tau_M)|^{p+1} \right) .$$

In view of (A.36) and of the definition of τ_M, we find an estimate for $|\tau_M|^p \tau_M$ in $L^2(0,T;V)$, again with some constant C_μ which is independent of $M > 0$. Hence, using the fact that $V \subset L^6(\Omega)$ and the growth condition (H1) for α, we conclude that

$$\|\alpha(\tau_M)\|_{L^{\rho_p}(0,T;L^{\kappa_p}(\Omega))} \le C_\mu \quad \text{for all } M > 0 \tag{A.39}$$

(where the indexes ρ_p and κ_p are as in (3.6): in particular, $1 < \rho_p < 2$). Therefore, we have

$$\int_0^T \int_\Omega |\alpha_M(w_{M,\mu})|^{\rho_p} \le \iint_{\mathcal{A}_M} |\alpha(\tau_M)|^{\rho_p} + 2^{\rho_p-1} \iint_{\mathcal{O}_M} |\alpha(M)|^{\rho_p} + 2^{\rho_p-1} C_1^{\rho_p} \iint_{\mathcal{O}_M} |w_{M,\mu} - M|^{\rho_p}$$

$$\le 2^{\rho_p-1} \int_0^T \int_\Omega |\alpha(\tau_M)|^{\rho_p} + C\|w_{M,\mu}\|^{\rho_p}_{L^{\rho_p}(0,T;L^{\rho_p}(\Omega))} + CM^{\rho_p}|\mathcal{O}_M|$$

$$\le C_\mu + C + C\frac{M^{\rho_p}}{M^{p+2}},$$

where the first inequality follows from the very definition (A.3) of α_M, the second one from trivial calculations, and the last one from estimates (A.23) for $w_{M,\mu}$, (A.38) for $|\mathcal{O}_M|$, and (A.39) for $\alpha(\tau_M)$. Note that, since $\rho_p < 2$, we have $M^{\rho_p}/M^{p+2} \to 0$ as $M \to +\infty$.

Altogether, we find

$$\|\alpha_M(w_{M,\mu})\|_{L^{\rho_p}(0,T;L^{\rho_p}(\Omega))} \le C_\mu \quad \text{for all } M > 0, \tag{A.40}$$

which yields, by comparison in (A.30),

$$\|\partial_t \chi_{M,\mu}\|_{L^{\rho_p}(0,T;W^{-2,\rho_p}(\Omega))} \le C_\mu \quad \text{for all } M > 0, \tag{A.41}$$

$W^{-2,\rho_p}(\Omega)$ denoting here the standard negative order Sobolev space.

Collecting estimates (A.23), (A.24), (A.32), and (A.40)–(A.41), we then argue in the same way as in Section 4.2. Thus, we conclude that there exist a subsequence $M_k \nearrow +\infty$ as $k \to +\infty$ (whose dependence on the index $\mu > 0$ is not highlighted) and functions (χ_μ, w_μ) fulfilling (3.13)–(3.14) such that the functions $(\chi_{M_k,\mu}, w_{M_k,\mu})$ converge, as $k \to +\infty$, to (χ_μ, w_μ) in the same sense as in (4.29)-(4.30) and (4.32), while, in place of (4.31), we only have

$$\partial_t \chi_{M_k,\mu} \rightharpoonup \partial_t \chi_\mu \quad \text{in } L^{\rho_p}(0,T;W^{-2,\rho_p}(\Omega)),$$

which is, anyway, sufficient for what follows. Furthermore, there exists some $\bar{\alpha}_\mu \in L^{\rho_p}(0,T;L^{\rho_p}(\Omega))$ such that

$$\alpha_M(w_{M_k,\mu}) \rightharpoonup \bar{\alpha}_\mu \quad \text{in } L^{\rho_p}(0,T;L^{\rho_p}(\Omega)).$$

Again, we prove that

$$\phi_\mu(\chi_{M_k,\mu}) \to \phi_\mu(\chi_\mu) \quad \text{in } L^2(0,T;H) \tag{A.42}$$

and, proceeding as in Section 4.2, with (A.42) we show that $\bar{\alpha}_\mu = \alpha(w_\mu)$ and

$$\alpha_{M_k}(w_{M_k,\mu}) \to \alpha(w_\mu) \quad \text{in } L^1(0,T;L^1(\Omega)).$$

Having this, we conclude that the pair (χ_μ, w_μ) is solution to the PDE system

$$\chi_t + A(\alpha(w)) = 0 \quad \text{a.e. in } \Omega \times (0,T), \tag{A.43}$$

$$A\chi + \phi_\mu(\chi) = w \quad \text{a.e. in } \Omega \times (0,T), \tag{A.44}$$

supplemented with the initial condition (A.12).

Third step. Finally, we take the limit $\mu \searrow 0$ in (the Cauchy problem for) (A.43)–(A.44). Estimate (A.23), with α_M replaced by α, holds for the sequence $\{(\chi_\mu, w_\mu)\}_\mu$ for a constant $C > 0$ which is *independent* of the parameter $\mu > 0$.

Furthermore, using the fact that system (A.43)–(A.44) has the same structure as (3.18)–(3.19), we argue as in (4.14)–(4.15) and conclude that

$$\exists\, C > 0 \ \ \forall\, \mu > 0 : \quad \|\chi_\mu\|_{L^2(0,T;W^{2,6}(\Omega))} + \|\phi_\mu(\chi_\mu)\|_{L^2(0,T;L^6(\Omega))} \leq C.$$

From the bound for $(\alpha'(w_\mu))^{1/2} \nabla w_\mu$ in $L^2(0,T;H)$ (which follows from (A.33) by applying once more Ioffe's theorem), developing the very same calculations as throughout (4.19)–(4.24b), we find

$$\begin{aligned}
\exists\, C > 0 \ \ \forall\, \mu > 0 : \quad \|\partial_t \chi_\mu\|_{L^{\eta_{p\sigma}}(0,T;\mathcal{W}^{-2,\kappa_p}(\Omega))} + \|\alpha(w_\mu)\|_{L^{\eta_{p\sigma}}(0,T;L^{\kappa_p}(\Omega))} \\
+ \|\phi_\mu(\chi_\mu)\|_{L^{\sigma q_\sigma}(0,T;L^6(\Omega))} \leq C
\end{aligned} \tag{A.45}$$

(where the indexes $\eta_{p\sigma}$ and q_σ are as in (3.6) and (4.20), respectively).

Thanks to the above estimates, we conclude that there exist a vanishing sequence $\mu_k \searrow 0$ and functions (χ, w) satisfying (3.13)–(3.14) such that $(\chi_{\mu_k}, w_{\mu_k})$ converges to (χ, w) in the topologies of (4.29)–(4.33). We then pass to the limit as $k \to +\infty$ in (A.11) and, also in view of (A.12), infer that χ complies with the initial condition (3.15). Furthermore, we deduce from the strong convergence of χ_{μ_k} to χ in $L^2(0,T;H)$ that $\chi_{\mu_k} \to \chi$ almost everywhere in $\Omega \times (0,T)$. Using the uniform convergence (A.5) of $\{\phi_{\mu_k}\}$ to ϕ, we infer that

$$\phi_{\mu_k}(\chi_{\mu_k}(x,t)) \to \phi(\chi(x,t)) \quad \text{for a.a. } (x,t) \in \Omega \times (0,T).$$

Then, taking into account the uniform integrability of $\{\phi_\mu(\chi_{\mu_k})\}$ in $L^2(0,T;H)$ (which follows from (A.45), noting that $\sigma q_\sigma > 2$), in view of Theorem 2.1 we obtain

$$\phi_{\mu_k}(\chi_{\mu_k}) \to \phi(\chi) \quad \text{in } L^2(0,T;H). \tag{A.46}$$

Then, we again argue as in Section 4.2 and use (A.46) to prove that

$$\alpha(w_{\mu_k}) \to \alpha(w) \quad \text{in } L^1(0,T;L^1(\Omega)).$$

Having this, we conclude that the pair (χ, w) is solution to Problem 1, which finishes the proof. $\qquad\square$

References

[1] A.V. Babin and M.I. Vishik: *Attractors of evolution equations*. Translated and revised from the 1989 Russian original by Babin. Studies in Mathematics and its Applications, 25. North-Holland Publishing Co., Amsterdam, 1992.

[2] J.M. Ball: Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equations. *J. Nonlinear Sci.*, **7** (1997), 475–502.

[3] V. Barbu: *Nonlinear Semigroups and Differential Equations in Banach Spaces*. Noordhoff, Leyden, 1976.

[4] V. Barbu, P. Colli, G. Gilardi, and M. Grasselli: Existence, uniqueness, and long-time behaviour for a nonlinear Volterra integrodifferential equation. *Differential Integral Equations*, **13** (2000), 1233–1262.

[5] H. Brézis: *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert*. North Holland Math. Studies, 5. North-Holland, Amsterdam, 1973.

[6] J.W. Cahn: *On spinodal decomposition*. Acta Metall., **9** (1961), 795–801.

[7] J.W. Cahn and J.E. Hilliard: Free energy of a nonuniform system I. Interfacial free energy. *J. Chem. Phys.*, **2** (1958), 258–267.

[8] N. Dunford and J.T. Schwartz: *Linear Operators. Part I*. Interscience Publishers, New York, 1958.

[9] A. Eden, C. Foias, B. Nicolaenko, and R. Temam: *Exponential attractors for dissipative evolution equations*. Research in Applied Mathematics, 37. John-Wiley, New York, 1994.

[10] M. E. Gurtin: Generalized Ginzburg-Landau and Cahn-Hilliard equations based on a microforce balance. *Phys. D*, **92** (1996), 178–192.

[11] M. Hieber and J. Prüss: Heat kernels and maximal L^p-L^q estimates for parabolic evolution equations. *Comm. Partial Differential Equations*, **22** (1997), 1647–1669.

[12] A.D. Ioffe: On lower semicontinuity of integral functionals. I. *SIAM J. Control Optimization*, **15** (1977), 521–538.

[13] J. Málek and D. Pražák: Large time behavior via the method of l-trajectories. *J. Differential Equations*, **181** (2002), 243–279.

[14] A. Miranville and S. Zelik: Robust exponential attractors for Cahn-Hilliard type equations with singular potentials. *Math. Methods Appl. Sci.*, **27** (2004), 545–582.

[15] A. Miranville and S. Zelik: *Attractors for dissipative partial differential equations in bounded and unbounded domains.* In: Handbook of Differential Equations, Evolutionary Partial Differential Equations, C.M. Dafermos and M. Pokorny eds., Elsevier, Amsterdam, 2008.

[16] A. Novick-Cohen: *The Cahn-Hilliard equation.* In: Handbook of Differential Equations, Evolutionary Partial Differential Equations, C.M. Dafermos and M. Pokorny eds., Elsevier, Amsterdam, 2008.

[17] E. Rocca and G. Schimperna: Universal attractor for some singular phase transition systems. *Phys. D*, **192** (2004), 279–307.

[18] R. Rossi: On two classes of generalized viscous Cahn-Hilliard equations. *Commun. Pure Appl. Anal.*, **4** (2005), 405–430.

[19] R. Rossi: Global attractor for the weak solutions of a class of viscous Cahn-Hilliard equations. In: *Dissipative Phase Transitions*, pp. 247–268. Series on Advances in Mathematics for Applied Sciences, Vol. 71, World Sci. Publ., Hackensack, NJ, 2006.

[20] R. Rossi, A. Segatti, and U. Stefanelli: Attractors for gradient flows of non convex functionals and applications to quasistationary phase field models. *Arch. Ration. Mech. Anal.*, **187** (2008), 91–135.

[21] G. Schimperna: Global attractors for Cahn-Hilliard equations with nonconstant mobility. *Nonlinearity*, **20** (2007), 2365–2387.

[22] A. Segatti: On the hyperbolic relaxation of the Cahn-Hilliard equation in 3D: approximation and long time behaviour. *Math. Models Methods Appl. Sci.*, **17** (2007), 411–437.

[23] J. Simon: Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, **146** (1987), 65–96.

[24] R. Temam: *Infinite-dimensional dynamical systems in mechanics and physics*, Second edition, Applied Mathematical Sciences, Vol. 68, Springer-Verlag, New York, 1997.